Chapter 2 Systems

2.1 The concept of a system

A **system** transforms an input into an output. The input and output may be functions which are usually thought of as depending on time, but can depend on space coordinates or any other "independent" variable(s). The values that the input and output can take may be real or complex and they can be scalars or vectors. A vector-valued input or output may sometimes simply be a mathematical representation of several inputs or outputs.

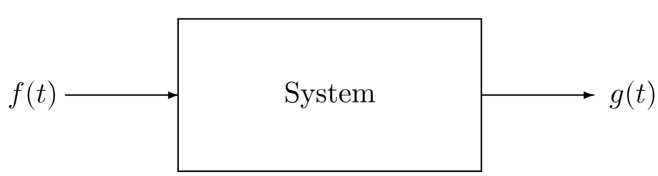
Examples of systems are

1. The input is the voltage applied to an electrical network as a function of time and the output is the voltage across some component in the network, which is also a function of time.
2. The input is the distribution of charge density in space (a scalar-valued function of the space coordinates) and the output is the distribution of electric field (a vector-valued function of the space coordinates).
3. The input is the position of the steering wheel, brake and accelerator pedals of a car (a three component vector function of time) and the the output is position of the car (a three component vector function of time).
4. The input is the wave function $\psi(\mathbf{r}, 0)$ of a particle at time $t = 0$ and the output is the wave function $\psi(\mathbf{r}, T)$ at some later time $t = T$. In this case both input and output are complex-valued functions of space.
5. The input is the distribution of light and shade of a scene and the output is the distribution of silver grains on a photographic negative taken of the scene through some imaging system (such as a camera). If the scene and image are divided into discrete picture elements, the input and output are large vectors which are related by the physics of the imaging process.

We shall mainly be concerned in the first part of this course with systems with scalar inputs and scalar outputs which are functions of a single independent variable which will usually be time. However, many of the methods developed are useful for more general systems as well.

2.2 Classes of systems

Consider a scalar-input scalar-output system with input $f(t)$ and output $g(t)$. In general, the output function g can depend on the entire input function f (i.e., its values at all times).



1. The system is said to be **causal** if $g(t)$ does not depend on the values of $f(\tau)$ for $\tau > t$.
2. The system is said to be **linear** if the output due to a linear combination of inputs is the same linear combination of the outputs due to each of the inputs separately. That is, if inputs $f_1(t)$ and $f_2(t)$ produce outputs $g_1(t)$ and $g_2(t)$ respectively, the input $c_1 f_1(t) + c_2 f_2(t)$ produces the output $c_1 g_1(t) + c_2 g_2(t)$ for all constants c_1 and c_2.

   Linear systems satisfy the **superposition principle** which is simply the above with $c_1 = c_2 = 1$. The importance of linearity is that it allows us to write the input as a sum of simpler inputs and to consider the effect of the system on each component separately. In particular, if we can write the input as a sum of the eigenfunctions of the linear system, the output is easily found since the effect of a system on an eigenfunction is simply to multiply it be the corresponding eigenvalue.
3. The system is said to be **memoryless** if the output at time t depends only on t and on the value of the input at time t (and not on the value of f at any other time).
4. The system is said to be **shift-invariant** or **time-invariant** if the response to a time-shifted version of the original input is the original output shifted by the same time. That is if $f(t)$ produces $g(t)$, then $f(t-\tau)$ produces $g(t-\tau)$.

2.2.1 Characterization of linear systems

Even though many practical systems are non-linear, it is often possible to linearize their characteristics for sufficiently small fluctuations about an operating point.

Example: Consider the relationship between pressure and density in a gas. Under adiabatic conditions $P \propto \rho^\gamma$ which is a non-linear relationship. However when we consider sound propagation for which the pressure and density fluctuations about the steady-state values are small, we find

$$(P_0 + \Delta P) = k(\rho_0 + \Delta\rho)^\gamma \tag{2.1}$$

$$\approx k\rho_0^\gamma + (k\gamma\rho_0^{\gamma-1})\,\Delta\rho \tag{2.2}$$

This leads to the following linear (and memoryless) relationship between ΔP and $\Delta\rho$

$$\Delta P = (k\gamma\rho_0^{\gamma-1})\,\Delta\rho \tag{2.3}$$

We can thus often apply linear systems theory as an approximation to the actual behaviour.

- The output of a **linear** system for an arbitrary input may be determined if we know how it responds to an impulse at a given time. This is based on the same argument as we used for Green's functions and differential equations.
  - Since any input $f(t)$ can be written as a linear combination of shifted delta functions,

    $$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\;\mathrm{d}\tau \tag{2.4}$$

  - If the input $\delta(t-\tau)$ produces the output $h(t|\tau)$ which is called the **impulse response** of the system at t to an impulse at τ,
  - then the output $g(t)$ must be the same linear combination of the impulse responses, namely

    $$g(t) = \int_{-\infty}^{\infty} f(\tau)\,h(t|\tau)\;\mathrm{d}\tau \tag{2.5}$$

- If the system is **linear** and **causal**, then $h(t|\tau) = 0$ for $t < \tau$.
- If the system is **linear** and **memoryless**, then $h(t|\tau) = c(t)\delta(t-\tau)$, and so $g(t) = c(t)f(t)$.
- If the system is **linear** and **time-invariant**, then the response to a delta function at $t-\tau$ is a shifted version of the response to a delta function at $t = 0$. i.e., $h(t|\tau) = h(t-\tau|0)$. For time-invariant systems we often write $h(t-\tau)$ in place of $h(t-\tau|0)$ and so the output is

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\ \mathrm{d}\tau \tag{2.6}$$

  This is called the **convolution** of f and h and is denoted by

$$g = f * h \tag{2.7}$$

  Note that this is a relationship between **functions**. It is often (loosely) written as $g(t) = f(t) * h(t)$ but this can be confusing since the value of g at time t depends on **all** the values of f and h, not only their values at time t. In this case $h(t)$ is the response to an impulse at the origin, which by time-invariance allows us to know how the system responds to an impulse located at any other time.
- If the system is **linear, memoryless** and **time-invariant** we find that $h(t) = c\delta(t)$ and so $g(t) = cf(t)$ for some constant c.

2.2.2 Graphical Interpretation

The result for a linear time-invariant system is of particular importance since many systems encountered in physics have these properties. Here we shall illustrate graphically the significance of the derivation given above (see Figure 2.1).

Instead of using delta functions, let us first fix on a (large, positive) integer N and define

$$\delta_N(t) = \begin{cases} N & \text{if } |t| < 1/(2N), \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

This function is of unit area and is concentrated on the interval $[-1/(2N), 1/(2N)]$ around $t = 0$. It is easy to check that δ_N tends distributionally to δ as $N \to \infty$.

Let us now consider a function $f(t)$ which we wish to apply as the input to our system. If we consider the function values at the times $\{k\Delta t\}_{k=-\infty}^{\infty}$ where $\Delta t \equiv 1/N$ is the width of the support of δ_N, and define

$$f_N(t) = \sum_{k=-\infty}^{\infty} \frac{1}{N} f\left(\frac{k}{N}\right) \delta_N\left(t - \frac{k}{N}\right) \equiv \sum_{k=-\infty}^{\infty} \left[\Delta t\, f(k\Delta t)\right] \delta_N(t - k\Delta t), \tag{2.9}$$

it is not difficult to see that $f_N(t)$ is a staircase approximation to the function $f(t)$. You should be able to show that if $f_N(t)$ is locally integrable in the Riemann sense, the distributional limit of f_N as $N \to \infty$ is just f. (i.e., show that given any test function $\phi \in \mathcal{D}$, $\langle f_N, \phi\rangle \to \langle f, \phi\rangle$ as $N \to \infty$.)

We wish to interpret the representation of f_N given by the above expression as being a sum of the **"basis" functions** $\delta_N(t - k\Delta t)$ weighted by the **coefficients** $\Delta t\, f(k\Delta t)$. If we introduce f_N into our linear time-invariant system, by **linearity**, the output may be found once we know how each component of the sum is transformed by the system.

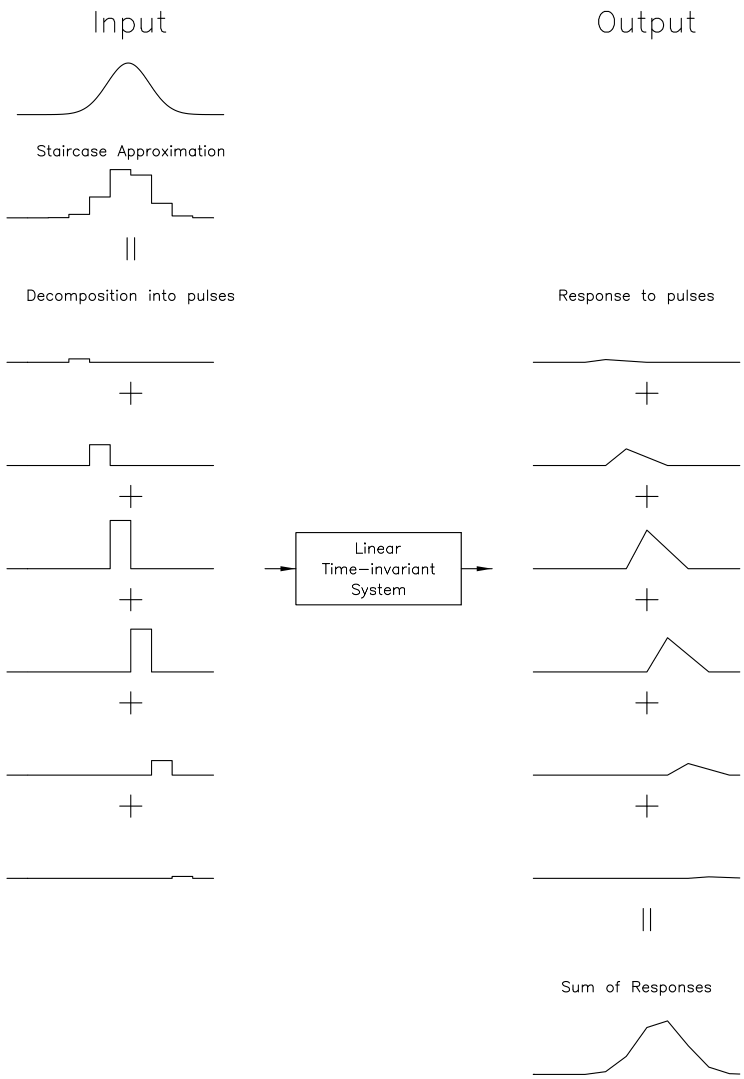


Figure 2.1 Response of a linear time-invariant system to an input signal expressed as a sum of responses to scaled and shifted pulses. In the continuum limit, the pulses turn into Dirac delta functions and responses become shifted impulse responses.

Suppose that the system responds to an input of $\delta_N(t)$ by producing the output $h_N(t)$. By **time-invariance**, the response to the input $\delta_N(t - k\Delta t)$ must simply be $h_N(t - k\Delta t)$. We now know how the system responds to all the basis functions. By linearity, when the input is $f_N(t)$, the output is the linear combination of the responses $h_N(t - k\Delta t)$ with the same coefficients as in the expansion of f_N in terms of the basis functions. The output is thus

$$\sum_{k=-\infty}^{\infty} \left[\Delta t\, f(k\Delta t)\right] h_N(t - k\Delta t). \tag{2.10}$$

If we now consider the distributional limit as $N \to \infty$, we see that this is the convolution integral

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau. \tag{2.11}$$

We thus regard this as a linear combination of delayed impulse responses $h(t-\tau)$ with weights $f(\tau)\,\mathrm{d}\tau$, just as $f(t)$ may be regarded as a linear combination of delayed impulses $\delta(t-\tau)$ with the same weights.

Technical note: If we wish to be more general and consider locally integrable functions $f(t)$ in the Lebesgue sense, we would define instead

$$f_N(t) = \sum_{k=-\infty}^{\infty} \left[\int_{(k-\frac{1}{2})\Delta t}^{(k+\frac{1}{2})\Delta t} f(\tau)\, \mathrm{d}\tau \right] \delta_N(t - k\Delta t) \tag{2.12}$$

where the integral is a Lebesgue integral. It is easy to check that f_N converges distributionally to $f(t)$ as $N \to \infty$.

Example: Direct Computation of a Convolution Integral

Consider the convolution of the functions

$$f(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(t) = \begin{cases} 1 - t & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

which are shown in Figure.2.2.

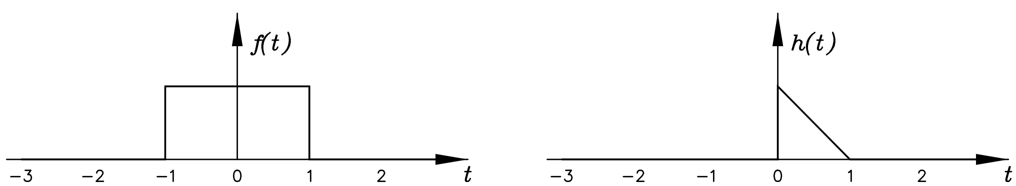


Figure 2.2 Functions to be convolved

By definition, the convolution of these functions is

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t - \tau)\, \mathrm{d}\tau.$$

Most of the complexity in this problem arises from having to consider the limits in the integral carefully. First we note that the integration is carried out as a function of τ and that it is useful to plot the integrand as a function of τ. The first factor $f(\tau)$ is straightforward, but when regarded as a function of τ, $h(t - \tau)$ involves reflecting the graph of h about the vertical axis and shifting the result by t. Depending on the value of t, the functions $f(\tau)$ and $h(t - \tau)$ overlap in different ways. We can identify five separate cases as shown in Figure 2.3(a)–(e).

Case (a) $t < -1$

Refering to Figure 2.3(a), we see that when $t < -1$, the product $f(\tau)\, h(t - \tau)$ is identically zero for all τ. The convolution integral $(f * h)(t)$ is thus equal to zero.

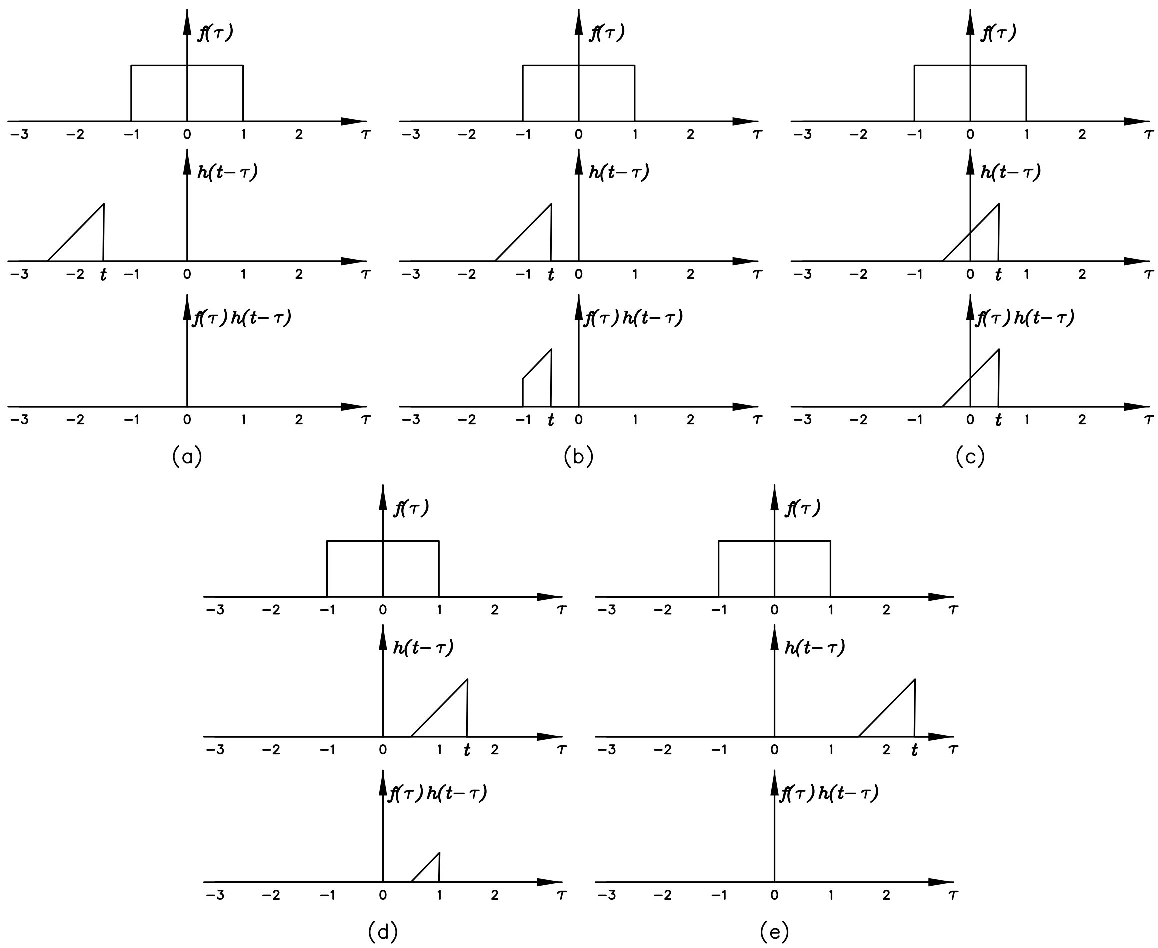


Figure 2.3 Computation of integrand of the convolution integral when (a) $t < -1$, (b) $-1 \leq t < 0$, (c) $0 \leq t < 1$, (d) $1 \leq t < 2$ and (e) $t \geq 2$.

Case (b) $-1 \leq t < 0$

Refering to Figure 2.3(b), we see that when $-1 \leq t < 0$, the product $f(\tau)\, h(t-\tau)$ is non-zero only over the interval $\tau \in [-1, t]$. Thus the limits of the convolution integral need to be set to these values

$$\begin{aligned}(f * h)(t) &= \int_{-1}^{t} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau = \int_{-1}^{t} 1 \times [1 - (t-\tau)]\, \mathrm{d}\tau \\ &= \int_{-1}^{t} (1 - t + \tau)\, \mathrm{d}\tau = \frac{1-t^2}{2}\end{aligned}$$

Case (c) $0 \leq t < 1$

Refering to Figure 2.3(c), we see that when $0 \leq t < 1$, the product $f(\tau)\, h(t-\tau)$ is non-zero over the interval $\tau \in [t-1, t]$. Thus the limits of the convolution integral need to be set to these values

$$\begin{aligned}(f * h)(t) &= \int_{t-1}^{t} f(\tau)\, h(t-\tau)\ \mathrm{d}\tau = \int_{t-1}^{t} 1 \times [1-(t-\tau)]\ \mathrm{d}\tau \\ &= \int_{t-1}^{t} (1-t+\tau)\ \mathrm{d}\tau = \frac{1}{2}\end{aligned}$$

Case (d) $1 \leq t < 2$

Refering to Figure 2.3(d), we see that when $1 \leq t < 2$, the product $f(\tau)\, h(t-\tau)$ is non-zero over the interval $\tau \in [t-1, 1]$. Thus the limits of the convolution integral need to be set to these values

$$\begin{aligned}(f * h)(t) &= \int_{t-1}^{1} f(\tau)\, h(t-\tau)\ \mathrm{d}\tau = \int_{t-1}^{1} 1 \times [1-(t-\tau)]\ \mathrm{d}\tau \\ &= \int_{t-1}^{1} (1-t+\tau)\ \mathrm{d}\tau = \frac{(t-2)^2}{2}\end{aligned}$$

Case (e) $t \geq 2$

Refering to Figure 2.3(e), we see that when $t < -1$, the product $f(\tau)\, h(t-\tau)$ is identically zero for all τ. The convolution integral $(f * h)(t)$ is thus equal to zero.

Summarizing the results of all the above cases, we see that the convolution integral is equal to

$$(f * h)(t) = \begin{cases} \dfrac{1-t^2}{2} & \text{for } -1 \leq t < 0 \\ \frac{1}{2} & \text{for } 0 \leq t < 1 \\ \dfrac{(t-2)^2}{2} & \text{for } 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

A graph of this function is shown in Figure 2.4.

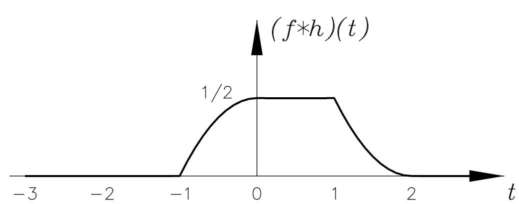


Figure 2.4 Result of convolution

2.3 Properties of the Convolution Integral

In the following, we regard f, g and h to be functions of time. You should be able to fill in the proofs of these from the definition of the convolution.

Technical note: We shall also assume that these results hold for **generalized functions** as well, although this is not always true since the convolution of two generalized functions is not always a generalized function. (e.g., $u(t) * u(-t)$ is undefined). It can be shown however that it is possible to define convolution of two generalized functions if one of them is of **bounded support** or if both of them have supports **bounded to the left** or if both of them have supports **bounded to the right**. A similar problem arises if we try to multiply together arbitrary generalized functions as the result is not always a generalized function (e.g., $\delta(t)^2$ is undefined). We shall not consider these issues in further detail, although many of the technical mathematical issues associated with the theory of distributions are precisely in this area.

2.3.1 Commutativity

$$f * g = g * f \tag{2.13}$$

Proof: By definition

$$f * g = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, \mathrm{d}\tau$$

Change variable of integration to $u = t - \tau$ and the result follows.

Thus the output of a linear time-invariant system with impulse response $h(t)$ for an input $f(t)$ is the same as that of a linear time-invariant system with impulse response $f(t)$ for an input $h(t)$.

2.3.2 Associativity

$$(f * g) * h = f * (g * h) \tag{2.14}$$

provided that the following (triple convolution) integral exists (and is finite)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(v-u) h(t-v)\, \mathrm{d}u\, \mathrm{d}v \tag{2.15}$$

Proof: Do as an exercise, using Fubini's theorem (i.e., double integrals can be evaluated as iterated integrals in either order).

It is thus possible to write $f * g * h$ unambiguously.

(*Technical note:* Counterexample showing that associativity is false if the triple convolution integral diverges.

$$(u * \delta') * 1 = \delta * 1 = 1$$

$$u * (\delta' * 1) = u * 0 = 0$$

The triple convolution integral diverges because $u * 1$ is divergent. Associativity is guaranteed if at most one of the terms in the multiple convolution vanishes outside some bounded interval.)

2.3.3 Linearity

For any constants c_1 and c_2,

$$(c_1 f_1 + c_2 f_2) * g = c_1 (f_1 * g) + c_2 (f_2 * g) \tag{2.16}$$

$$f * (c_1 g_1 + c_2 g_2) = c_1 (f * g_1) + c_2 (f * g_2) \tag{2.17}$$

Proof: This follows from linearity of the integral.

2.3.4 Shifting

Suppose that we have a system with the property that whenever a signal $f(t)$ is fed in, the output $g(t)$ is a delayed version of the input, i.e.,

$$g(t) = f(t - \tau) \tag{2.18}$$

where τ is the delay time. It is easy to see that such a system is both linear and shift-invariant. By our fundamental result, it must be possible to characterize this system completely by specifying its impulse response $h(t)$. In order to find the impulse response, we need only ask what the output is when an impulse $\delta(t)$ is fed into the system. Since the system delays any input by τ, the output for the impulsive input must be $h(t) = \delta(t-\tau)$ which we may denote by $\delta_\tau(t)$.

We now expect to be able to calculate the response to an arbitrary input by means of the convolution. If the input is $f(t)$, the output will be $(f * \delta_\tau)$ which should be equal to $f(t-\tau)$ by the above argument. Thus,

$$(f * \delta_\tau)(t) = f(t - \tau) \tag{2.19}$$

which is sometimes loosely written as $f(t) * \delta(t-\tau) = f(t-\tau)$.

You should check that you can prove this result formally as well, using the definition of the delta function.

Exercise: Convince yourself that for any function f,

$$(f * \delta)(t) = f(t), \tag{2.20}$$

i.e., if one builds a linear, shift-invariant system whose response to an impulse $\delta(t)$ is simply to produce the output $\delta(t)$, then its output is always equal to its input.

2.3.5 Relationship with differentiation

$$(f * g)' = f' * g = f * g' \tag{2.21}$$

Proof:

$$(f * g)' = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} f(\tau) g(t-\tau)\,\mathrm{d}\tau \tag{2.22}$$

$$= \int_{-\infty}^{\infty} f(\tau) \frac{\partial}{\partial t} g(t-\tau)\,\mathrm{d}\tau \tag{2.23}$$

$$= \int_{-\infty}^{\infty} f(\tau) g'(t-\tau)\,\mathrm{d}\tau \tag{2.24}$$

$$= f * g' \tag{2.25}$$

The other equality follows from commutativity.

Repeated application of this result yields

$$(f * g)^{(m+n)} = f^{(m)} * g^{(n)} \tag{2.26}$$

where $f^{(m)}$ denotes the m'th derivative of f, etc. Note that this result applies for generalized functions which are always infinitely differentiable.

A somewhat more physical way of proving this result involves considering a system which is a differentiator. Such a system simply produces an output which is the derivative of its input. It is easy to see that the differentiator is a linear and shift-invariant system so its action on any input f must be the convolution f and the impulse response of the system. Since the response of a differentiator to $\delta(t)$ is simply to produce $\delta'(t)$, it follows that $\delta'(t)$ is the impulse response, and hence that

$$f' = f * \delta' \tag{2.27}$$

for any function f. The required result then follows straightforwardly from the associativity of the convolution.

2.3.6 Relationship with integration

If

$$F(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau, \quad \text{and} \tag{2.28}$$

$$G(t) = \int_{-\infty}^{t} g(\tau)\,\mathrm{d}\tau \tag{2.29}$$

then

$$(F * g)(t) = (f * G)(t) = \int_{-\infty}^{t} (f * g)(\tau)\,\mathrm{d}\tau \tag{2.30}$$

Proof: Do as an exercise. Start from the definition and be careful about the integration limits.

Alternatively we may regard the definite integral of f from $-\infty$ to t as the convolution of f with a unit step. i.e.,

$$F(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau = f * u \tag{2.31}$$

(to see this, just think about the impulse response of a system which integrates its input). From associativity and commutativity of the convolution, it follows that the integral of the convolution of two functions can be found by integrating either function and convolving with the other. i.e.,

$$(f * g) * u = (f * u) * g = f * (g * u) \tag{2.32}$$

Exercise: Show that $u * \delta' = \delta$. Hence show that if $F(t)$ is given by (2.31), $F * g' = f * g$.

2.4 Example: Indirect Computation of a Convolution Integral

Here we reconsider the problem of convolving

$$f(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(t) = \begin{cases} 1 - t & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

using some of the properties of the convolution integral discussed. In particular, we note that:

- Convolving any function with a delta function is a particularly simple operation due to the shifting property.

- It is possible to reduce a piecewise polynomial function to a collection of delta functions by differentiation.
- Convolution is a linear operation.

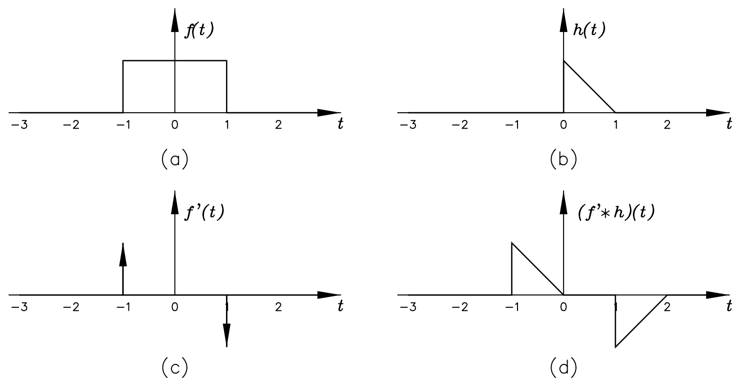


Figure 2.5 Indirect convolution of two functions f and h by first computing $f' * h$ and integrating the result.

- Differentiating $f(t)$ with respect to time reduces it to two delta functions (see Figure 2.5(c)):

$$f'(t) = \delta(t+1) - \delta(t-1) \tag{2.33}$$

Convolving f' with h can be done using the shifting property and linearity (see Figure 2.5(d)):

$$\left(f' * h\right)(t) = h(t+1) - h(t-1) \tag{2.34}$$

$$= \begin{cases} -t & \text{for } -1 \le t < 0 \\ t-2 & \text{for } 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.35}$$

This is the derivative of the desired convolution $f * h$. It thus remains to compute the integral, which may readily be found as follows:

$$\int_{-\infty}^{t} \left(f' * h\right)(\tau)\, \mathrm{d}\tau = \begin{cases} \int_{-1}^{t} -\tau\, \mathrm{d}\tau & \text{for } -1 \le t < 0 \\ \frac{1}{2} & \text{for } 0 \le t < 1 \\ \frac{1}{2} + \int_{1}^{t} (\tau - 2)\, \mathrm{d}\tau & \text{for } 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1-t^2}{2} & \text{for } -1 \le t < 0 \\ \frac{1}{2} & \text{for } 0 \le t < 1 \\ \dfrac{(t-2)^2}{2} & \text{for } 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

This is the same result as found earlier by direct evaluation of the convolution integral.

2.5 Stability of linear time-invariant systems

A linear system is said to be **stable** if the output is bounded for **every** bounded input.

i.e., if there exists $M > 0$ such that $|f(t)| < M$ for all time t, then when $f(t)$ is applied to the linear system, we must be able to find $K > 0$ such that $|g(t)| < K$ for all t.

Theorem: A linear time-invariant system is stable if and only if its impulse response $h(t)$ is absolutely integrable, i.e.,

$$I = \int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty \tag{2.37}$$

Proof: ($\Leftarrow$) We assume that the integral I is finite and have to show that the output for every bounded input is bounded.

Let the input $f(t)$ be bounded, so that there is some $M > 0$ such that $|f(t)| < M$ for all t. Then since

$$|g(t)| = \left| \int_{-\infty}^{\infty} f(t-\tau) h(\tau) \, \mathrm{d}\tau \right| \tag{2.38}$$

$$\leq \int_{-\infty}^{\infty} |f(t-\tau)| \, |h(\tau)| \, \mathrm{d}\tau \tag{2.39}$$

$$< M \int_{-\infty}^{\infty} |h(\tau)| \, \mathrm{d}\tau = MI \tag{2.40}$$

we see that $|g(t)|$ is bounded above by MI.

($\Rightarrow$) We now suppose that the integral for I diverges. We shall construct a bounded input function such that the output for this particular input diverges at $t = 0$. (i.e., we prove the contrapositive statement: I diverges $\Rightarrow$ system is not stable.)

Define the input function in terms of the impulse response,

$$f(t) = \begin{cases} |h(-t)| / h(-t) & \text{if } h(-t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.41}$$

This is clearly bounded since $|f(t)| \leq 1$ for all t. The output for this input is

$$g(t) = \int_{-\infty}^{\infty} f(t-\tau) h(\tau) \, \mathrm{d}\tau \tag{2.42}$$

$$= \int_{-\infty}^{\infty} \frac{|h(\tau - t)|}{h(\tau - t)} h(\tau) \, \mathrm{d}\tau \tag{2.43}$$

At $t = 0$, the output is

$$g(0) = \int_{-\infty}^{\infty} |h(\tau)| \, \mathrm{d}\tau \tag{2.44}$$

which is infinite, by assumption. The system is thus not stable.

Exercise: Is the system consisting of the series LCR circuit (where the input voltage is applied across all three components and the output voltage is taken across the capacitor) stable in the limit as $R \to 0$?

2.6 Eigenfunctions and eigenvalues of linear time-invariant systems

Linear time-invariant systems have the important property that whenever a complex exponential is applied as the input, the output is always a complex exponential of the same frequency. The amplitude of the output is in general a frequency-dependent complex number. Mathematically, it is no problem to consider feeding a complex input signal into a system, but it is also possible to interpret the operation physically by considering feeding the real and imaginary parts of the signal separately into two identical copies of the system. If the complex signal is written in terms of its real and imaginary parts $f(t) = f_r(t) + \mathrm{j}f_i(t)$, and if the outputs produced by inputs $f_r(t)$ and $f_i(t)$ are $g_r(t)$ and $g_i(t)$ respectively, then by linearity the response to $f(t)$ must be $g_r(t) + \mathrm{j}g_i(t)$.

Consider a linear time-invariant system with impulse response $h(t)$. If the input to the system is the complex exponential of frequency ν,

$$f(t) = \exp(\mathrm{j}2\pi\nu t), \tag{2.45}$$

then the output is given by the convolution

$$\begin{aligned}
g(t) &= (f * h)(t) = (h * f)(t) && (2.46)\\
&= \int h(\tau) f(t-\tau)\, \mathrm{d}\tau && (2.47)\\
&= \int h(\tau) \exp(\mathrm{j}2\pi\nu[t-\tau])\, \mathrm{d}\tau && (2.48)\\
&= \left(\int h(\tau) \exp(-\mathrm{j}2\pi\nu\tau)\, \mathrm{d}\tau\right) \exp(\mathrm{j}2\pi\nu t) && (2.49)\\
&= H(\nu) \exp(\mathrm{j}2\pi\nu t). && (2.50)
\end{aligned}$$

We see that $g(t)$ is simply a multiple of the input. $H(\nu)$ is called the **system function** or the **transfer function**. If we regard the system as a linear transformation which maps f into g, this means that $\exp(\mathrm{j}2\pi\nu t)$ is an eigenfunction of **any** linear, time-invariant system and the associated eigenvalue is $H(\nu)$. As we shall see, $H(\nu)$ is the Fourier transform of the impulse response $h(t)$.

This is the basis of the result used in a.c. circuit theory that whenever a linear (time-invariant) electrical network is excited with sinusoidal generators of a single frequency, all the currents and voltages in the network are sinusoidal with the same frequency. They only differ in their amplitudes and phases.

Exercise: Show that if $h(t)$ is real-valued and if $f(t) = \cos 2\pi\nu t$, then

$$g(t) = |H(\nu)| \cos(2\pi\nu t + \arg[H(\nu)])$$

so that the modulus of $H(\nu)$ gives the gain and $\arg[H(\nu)]$ gives the phase shift.

More generally, $\exp(st)$ for any complex s is an eigenfunction of a linear time-invariant system (provided that the output remains finite, which will usually be the case for at least some values of s). This can be demonstrated using only the definitions of linearity and time-invariance. Suppose we write the output of the linear time-invariant system when the input is $\exp(st)$ as $g(s,t)$.

If the input is changed to $\exp[s(t+\tau)]$, we can calculate the new output in one of two ways:

1. By time-invariance, the output must be given by $g(s, t+\tau)$, or

2. if we write $\exp[s(t+\tau)]$ as $\exp(s\tau)\exp(st)$, then by linearity, the output must be $\exp(s\tau)g(s,t)$.

These must be the same and so

$$g\left(s,t+\tau\right)=\exp(s\tau)g(s,t), \tag{2.51}$$

for every t and τ. If we set $t=0$, we find that

$$g\left(s,\tau\right)=\exp(s\tau)g(s,0), \tag{2.52}$$

and since τ is a dummy variable, we may relabel it as t to obtain

$$g\left(s,t\right)=g(s,0)\exp(st). \tag{2.53}$$

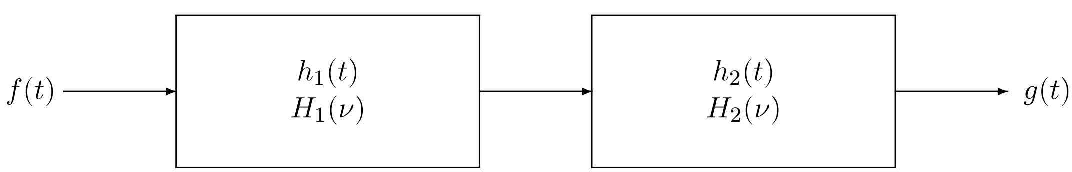
Thus the output when the input is $\exp\left(st\right)$ is the input multiplied by a quantity which is a function of s alone. This includes the previous result as a special case when $s=\mathrm{j}2\pi\nu$ is purely imaginary.

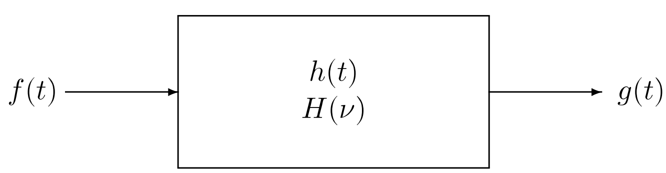
2.7 Cascaded systems

Two systems are said to be cascaded when the output of the first is used as the input to the second. If the impulse responses $h_1(t)$ and $h_2(t)$ of the two systems are known, it is easy to compute the impulse response $h(t)$ of the cascaded system.

Consider an input function f. The output of the first system is $f*h_1$ which is used as the input for the second system. The output of the second system is thus $(f*h_1)*h_2$. By the associativity of the convolution, we see that

$$h\left(t\right)=\left(h_1*h_2\right)\left(t\right). \tag{2.54}$$





If we now consider the corresponding transfer functions $H_1(\nu)$ and $H_2(\nu)$, we see that an input of $\exp(\mathrm{j}2\pi\nu t)$ yields an output from the first system of $H_1(\nu)\exp(\mathrm{j}2\pi\nu t)$ since complex exponentials are eigenfunctions of the first system. The second system in turn multiplies this complex exponential by $H_2(\nu)$ so that the output of the cascaded system is $H_1(\nu)H_2(\nu)\exp(\mathrm{j}2\pi\nu t)$. We conclude that complex exponentials are also eigenfunctions of the cascaded system and that the eigenvalue at frequency ν is given by

$$H\left(\nu\right)=H_1\left(\nu\right)H_2\left(\nu\right). \tag{2.55}$$

From these results, we see that convolution of the impulse responses corresponds to multiplication of the associated transfer functions. We shall return to this result in connection with Fourier transforms.

Exercises:

1. Calculate the convolution of

$$f(t) = \begin{cases} 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(t) = \begin{cases} \sin(\pi t) & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

2. Show that the convolution of $f(t) = \exp(-at^2)$ and $h(t) = \exp(-bt^2)$ is also a Gaussian. In fact

$$(f * h)(t) = \sqrt{\frac{\pi}{a+b}} \exp\left(-\frac{ab}{a+b}t^2\right)$$

which is wider than either of the original Gaussians.

3. Calculate the convolution of $\text{sinc}(t)$ with itself, directly from the definition of the convolution integral, where

$$\text{sinc}(t) = \begin{cases} \dfrac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

4. Calculate the convolution of $\text{sinc}(at)$ and $\text{sinc}(bt)$ from the definition. The result should be

$$\frac{\text{sinc}\left[\min\left(a,b\right)t\right]}{\max(a,b)}.$$