Chapter 1

Example problems for modelling in physics

We need to gain experience in setting up the equations which model physical systems as well as experience in solving the resulting equations. Proficiency in this subject is gained by studying many examples and practicing on as wide a variety of problems as possible. In this chapter, we consider a few general techniques which have been found to be useful in many situations. For further development of the ideas developed in these examples, refer to the book *Introduction to Applied Mathematics* by Gilbert Strang.

1.1 An inextensible string under tension

1.1.1 Equilibrium with a sideways distribution of force

Consider a light, inextensible string which lies along the x axis and which is under tension due to a force T. A sideways force per unit length $p(x)$ is applied to the string, which causes the string to take on the shape $y(x)$ when in equilibrium. We wish to find the relationship between $p(x)$ and $y(x)$.

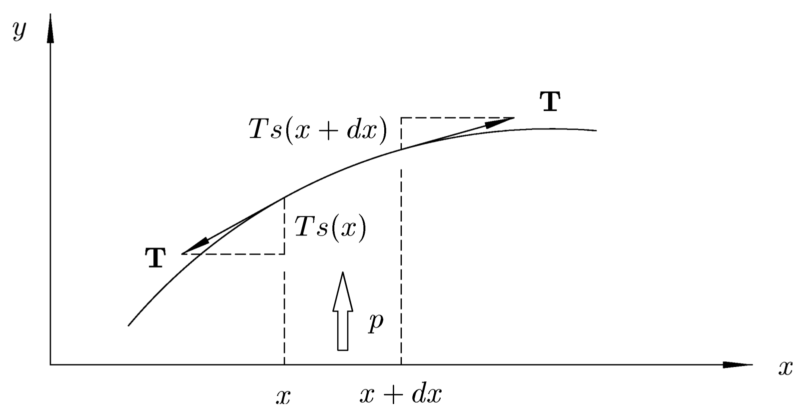

Figure 1.1: Transverse equilibrium of string with sideways force

Consider dividing the string into small elements of length $\mathrm{d}x$, and consider the equilibrium of each element. The external (applied) force on the element in the y direction is $p(x)\,\mathrm{d}x$.

The element is also acted on by internal forces due to the portions of the string on its right and left hand sides. Let the right-hand boundary of the element be at an angle $\theta(x + \mathrm{d}x)$ to the x axis. The tension in the string acting on that side of the element has a component in the y direction given by $T \sin[\theta(x + \mathrm{d}x)]$. If the angle θ is small, we may approximate $\theta \approx \sin\theta \approx \tan\theta$. Since $\tan\theta$ is just the slope of the equation of the string at x which may be obtained from the derivative of $y(x)$, we can define

$$s(x) = \frac{\mathrm{d}y}{\mathrm{d}x}, \tag{1.1}$$

and conclude that the vertical component of the tension on the right hand side of the element is $T \sin[\theta(x + \mathrm{d}x)] \approx T \tan[\theta(x + \mathrm{d}x)] = Ts(x + \mathrm{d}x)$. Similarly, if the left-hand boundary of the element is at angle $\theta(x)$ to the x axis, the tension in the string acting on that side of the element has a component in the y direction given by $-T \sin[\theta(x)] \approx -Ts(x)$. The total internal force in the y direction on the element is thus

$$Ts(x + \mathrm{d}x) - Ts(x). \tag{1.2}$$

If the string is to be in equilibrium, the sum of the internal forces and the external force on the element must vanish. i.e.,

$$p(x)\,\mathrm{d}x + Ts(x + \mathrm{d}x) - Ts(x) = 0. \tag{1.3}$$

By Taylor's theorem,

$$s(x + \mathrm{d}x) = s(x) + \frac{\mathrm{d}s}{\mathrm{d}x}\mathrm{d}x + \cdots \tag{1.4}$$

and so

$$s(x + \mathrm{d}x) - s(x) \approx \frac{\mathrm{d}s}{\mathrm{d}x}\mathrm{d}x = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\mathrm{d}x. \tag{1.5}$$

The differential equation of the string can thus be written as

$$p(x) + T\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0 \tag{1.6}$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{p}{T}. \tag{1.7}$$

This is a one-dimensional form of Poisson's equation. Together with this differential equation, we need boundary conditions, which for the string might be that the displacements at the end of the string are zero, i.e., $y(0) = y(L) = 0$.

The above derivation may be summarized by the diagram of Figure 1.2.

1.1.2 A transverse wave on the string

Consider the string once again, this time without the external applied force. If the ends of the string are at $y = 0$, the equilibrium position of the string is $y = 0$ for all x. Now consider what happens if the string is displaced from the equilibrium position and released. We now wish to find how y varies both with position x and time t. The equation of motion of the string may be found by applying Newton's second law to each element of length $\mathrm{d}x$. As in the previous derivation, the internal forces on the element in the y direction at time t sum to

$$Ts(x + \mathrm{d}x, t) - Ts(x, t),$$

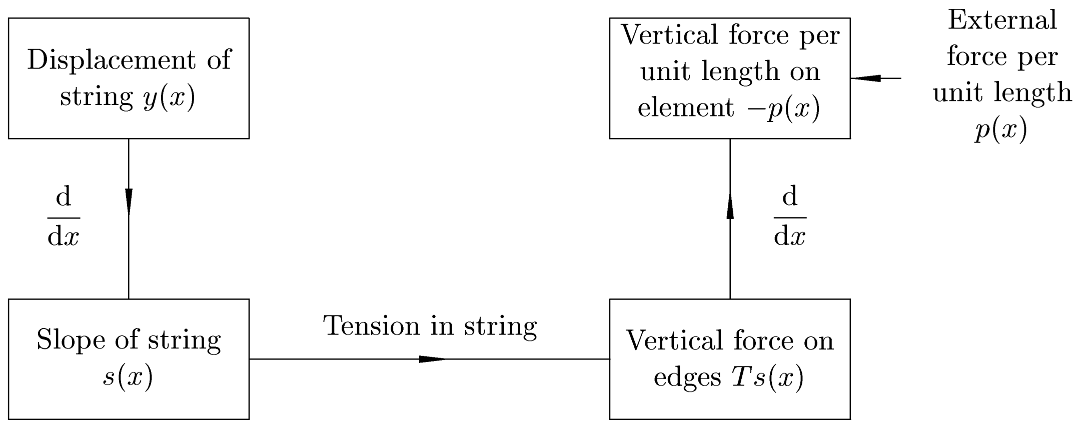

Figure 1.2: Transverse equilibrium of string in general framework

where

$$s(x,t) = \frac{\partial y}{\partial x}(x,t)\,. \tag{1.8}$$

Partial derivatives are used since y is a function of both x and t. If the linear density (mass per unit length) of the string is ρ, the mass of the element is $\rho \mathrm{d}x$ and Newton's second law for the element is

$$\rho \mathrm{d}x \frac{\partial^2 y}{\partial t^2}(x,t) = Ts(x+\mathrm{d}x,t) - Ts(x,t)\,. \tag{1.9}$$

Since s is the derivative of y in the x direction, we may use the argument leading to equation (1.5) to conclude that

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}, \tag{1.10}$$

which is the wave equation in one dimension for waves travelling at speed $\sqrt{T/\rho}$.

The above derivation may be summarized by the diagram of Figure 1.3.

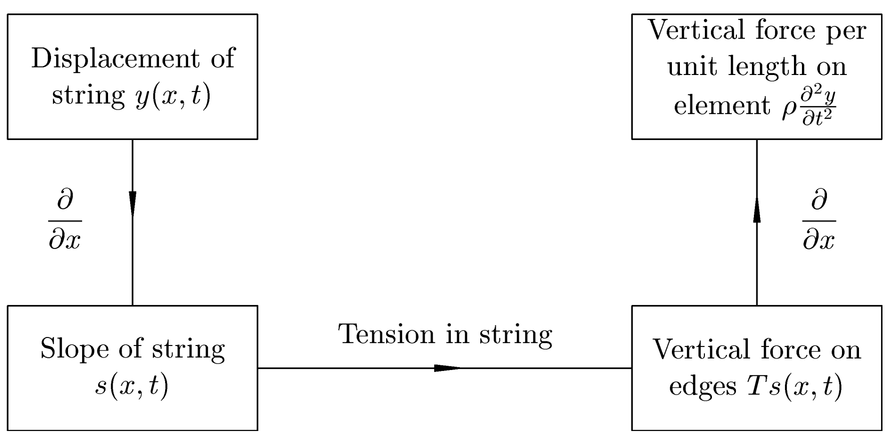

Figure 1.3: Motion of string in general framework

1.2 Heat diffusion in a one-dimensional rod

Consider a long rod along the x direction with cross sectional area A. Let $T(x,t)$ denote the temperature at the point x in the rod at time t. We wish to determine how the temperature distribution evolves in time due to heat diffusion. Let ρ be the density (mass per unit volume) of the rod, c be the heat capacity per unit mass and k be the therrmal conductivity, defined such that the energy (heat) flow per unit through an element of length ΔL and cross sectional area A due to a temperature difference of ΔT across the ends is $kA\Delta T/\Delta L$. We shall suppose that ρ, c and k are functions of x, the position along the rod.

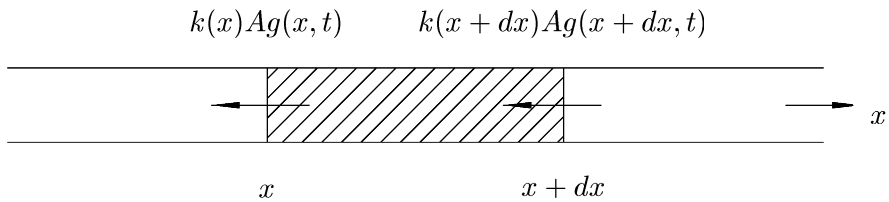

Figure 1.4: Heat flow along a rod

Consider an element at x of length $\mathrm{d}x$, which has mass $\rho A\mathrm{d}x$. The heat entering the element per unit time through the right hand boundary at $x+\mathrm{d}x$ depends on the temperature gradient $g \equiv \partial T/\partial x$ at that boundary and is given by

$$k(x+\mathrm{d}x)\,Ag(x+\mathrm{d}x,t) \tag{1.11}$$

Similarly, the heat leaving the element per unit time through the left hand boundary at x is

$$k(x)\,Ag(x,t) \tag{1.12}$$

The net flow of heat per unit time into the element is

$$k(x+\mathrm{d}x)\,Ag(x+\mathrm{d}x,t) - k(x)\,Ag(x,t) \approx A\frac{\partial}{\partial x}\left[k(x)\,g(x,t)\right]\mathrm{d}x \tag{1.13}$$

The change in the temperature of the element per unit time multiplied by the heat capacity is equal to this heat flow, i.e.,

$$\rho(x)\,c(x)\,A\mathrm{d}x\frac{\partial T}{\partial t} = A\frac{\partial}{\partial x}\left[k(x)\,g(x,t)\right]\mathrm{d}x, \tag{1.14}$$

since $\rho A\mathrm{d}x$ is the mass of the element and c is the heat capacity per unit mass. Thus,

$$\frac{\partial T}{\partial t} = \frac{1}{\rho(x)\,c(x)}\frac{\partial}{\partial x}\left(k(x)\frac{\partial T}{\partial x}\right). \tag{1.15}$$

In the general framework, shown in Figure 1.5, the basic physics of the diffusion process is contained in the law of thermal conduction, which states that the heat flow is proportional to the temperature gradient. This relationship is shown in the bottom part of the diagram. On the left hand side, we see how the temperature gradient is obtained from the temperature by using the spatial derivative operator $\partial/\partial x$. On the right-hand side the heat flows through the boundaries of an element are related to the net heat entering the element. This relationship again involves the spatial derivative $\partial/\partial x$. In the upper right-hand box, the rate at which heat enters the element is related to the change of temperature of the element. As we shall see in more detail later, the two vertical arms of the diagram are basically determined by the geometry of the problem, and these combine with the physics in the bottom line to give the governing equation.

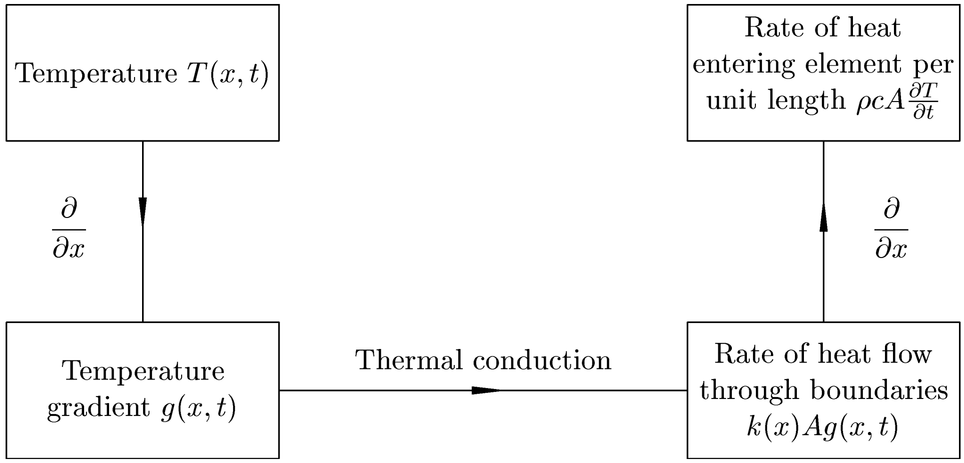

Figure 1.5: One dimensional heat diffusion in general framework

1.3 A longitudinal wave in a one-dimensional elastic medium

1.3.1 Mass and spring model

Consider a collection of masses each of mass m connected by springs with spring constant k. Let the positions of the masses be measured relative to their positions when all the springs have their natural lengths l. Thus, u_i denotes the displacement of the i'th mass from the reference position of that mass. We wish to consider the equations of motion of the masses (see Figure 1.6).

The displacements of all the masses may be collected into a vector $\mathbf{u}$ whose components are $u_1, u_2, \ldots, u_N$. From these displacements, we can evaluate the extensions of the springs. For the first spring which has one end fixed, the extension $e_1 = u_1$. Subsequent springs have extensions given by the differences of the displacements, so that $e_i = u_i - u_{i-1}$. We may write the relationship between extensions and displacements as

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \\ e_N \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} \tag{1.16}$$

or more briefly as $\mathbf{e} = \mathbf{A}\mathbf{u}$.

Given the extensions of the springs, Hooke's law states that the tensions in the springs are directly proportional to the extensions. If we denote the tension force in the i'th spring by w_i, we see that

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} = \begin{pmatrix} k & & & & \\ & k & & & \\ & & \ddots & & \\ & & & k & \\ & & & & k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \\ e_N \end{pmatrix} \tag{1.17}$$

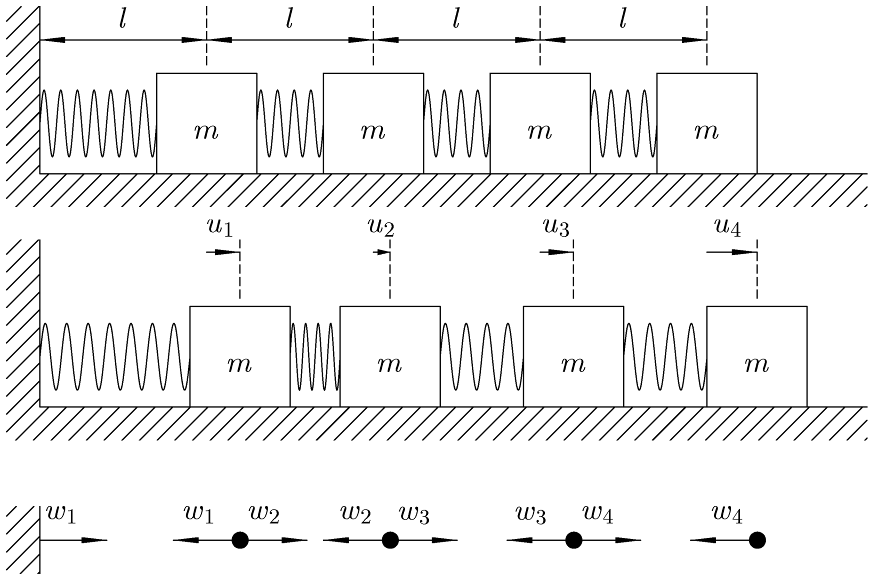

Figure 1.6: Mass and spring model of a one dimensional elastic medium. Upper figure shows equilibrium positions of masses, middle figure shows definition of displacements and lower figure shows forces on masses due to tensions in the springs

If we denote the diagonal matrix of spring constants by $\mathbf{C}$, this may be written as $\mathbf{w} = \mathbf{Ce}$. If all the spring constants are equal, we strictly do not have to write the relation in matrix form, but this general formalism may be readily extended to the situation of unequal spring constants.

Finally we can work out the net force on each mass and relate this to the acceleration of that mass. The i'th mass is attached to the $i+1$'th mass by spring $i+1$ and to the $i-1$'th mass by spring i. The net force f_i on mass i is equal to the difference between the tensions in these springs, i.e., $f_i = w_{i+1} - w_i$, or

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{pmatrix} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} \tag{1.18}$$

Due to the geometry of the problem, the matrix relating $\mathbf{f}$ and $\mathbf{w}$ is just $-\mathbf{A}^{\mathrm{t}}$ where $\mathbf{A}$ is the matrix which relates $\mathbf{e}$ to $\mathbf{u}$. We can thus write $\mathbf{f} = -\mathbf{A}^{\mathrm{t}}\mathbf{w}$, and Newton's second law becomes

$$m\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} = \mathbf{f} = -\mathbf{A}^{\mathrm{t}}\mathbf{w} = -\mathbf{A}^{\mathrm{t}}\mathbf{Ce} = -\mathbf{A}^{\mathrm{t}}\mathbf{CA}\,\mathbf{u} \tag{1.19}$$

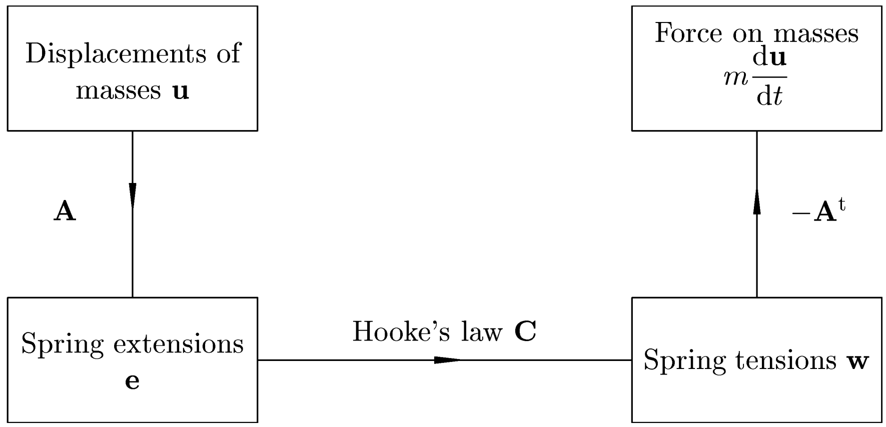

Figure 1.7: One dimensional elastic medium in general framework

Writing this out more fully,

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{pmatrix}\begin{pmatrix} k & & & & \\ & k & & & \\ & & \ddots & & \\ & & & k & \\ & & & & k \end{pmatrix} \times$$
$$\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} \tag{1.20}$$

or

$$m\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} = k\left(u_2 - 2u_1\right) \tag{1.21}$$
$$m\frac{\mathrm{d}^2 u_2}{\mathrm{d}t^2} = k\left(u_3 - 2u_2 + u_1\right) \tag{1.22}$$
$$m\frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = k\left(u_{i+1} - 2u_i + u_{i-1}\right) \tag{1.23}$$
$$m\frac{\mathrm{d}^2 u_N}{\mathrm{d}t^2} = k\left(u_N - u_{N-1}\right) \tag{1.24}$$

Let us now consider the limit in which the system of masses and springs turns into a continuous one-dimensional elastic medium. This can be done by letting m and l both go to zero in such a way that the ratio m/l tends to a constant. We also need to consider how the spring constant k should be adjusted in this limiting process. If we consider an elastic rod of cross section area A and length l, the tension w required to change its length by e is

$$w = EA\left(\frac{e}{l}\right)$$

where E is the Young's modulus of the material. The effective spring constant is thus $k = EA/l$. If the density of the elastic rod is ρ, the ratio m/l tends to ρA, so that $m = \rho Al$.

Once we consider the continuum limit, the displacement vector $\mathbf{u}(t)$ becomes a function of two variables, $u(x,t)$ where x is the reference position of the mass. Since the natural lengths of the springs are l, we should identify $u_i(t)$ with $u(il,t)$. The differential equation for u_i becomes

$$m\frac{\partial^2 u(il,t)}{\partial t^2} = k\{u([i+1]l,t) - 2u(il,t) + u([i-1]l,t)\} \tag{1.25}$$

We now use Taylor's theorem to expand the displacement around $x = il$ to second order:

$$u(il+\xi,t) \approx u(il,t) + \xi\frac{\partial u}{\partial x}(il,t) + \frac{1}{2}\xi^2\frac{\partial^2 u}{\partial x^2}(il,t) \tag{1.26}$$

Thus,

$$u([i+1]l,t) \approx u(il,t) + l\frac{\partial u}{\partial x}(il,t) + \frac{1}{2}l^2\frac{\partial^2 u}{\partial x^2}(il,t) \tag{1.27}$$

$$u([i-1]l,t) \approx u(il,t) - l\frac{\partial u}{\partial x}(il,t) + \frac{1}{2}l^2\frac{\partial^2 u}{\partial x^2}(il,t) \tag{1.28}$$

and so

$$u([i+1]l,t) - 2u(il,t) + u([i-1]l,t) \approx l^2\frac{\partial^2 u}{\partial x^2}(il,t). \tag{1.29}$$

The ordinary differential equation turns into a partial differential equation

$$m\frac{\partial^2 u(il,t)}{\partial t^2} = kl^2\frac{\partial^2 u}{\partial x^2}(il,t) \tag{1.30}$$

Taking il to be an arbitrary point in the medium, and writing m, k and l in terms of the bulk properties of the elastic material, we see that $kl^2/m = E/\rho$ and so

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{E}{\rho}\right)\frac{\partial^2 u}{\partial x^2} \tag{1.31}$$

which is a wave equation for waves of speed $\sqrt{E/\rho}$.

It is interesting to see how the boundary equations arise from the transition from the discrete to the continuous descriptions. At $x = 0$, where the spring is attached to the wall, we can introduce a new component u_0 to the displacement vector $\mathbf{u}$ with the understanding that it is always zero. The differential equation for u_1 can then be written as

$$m\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} = k(u_2 - 2u_1 + u_0) \tag{1.32}$$

which is exactly of the same form as the equations for the interior of the material. This boundary condition can thus be included by specifying that $u(0,t) = 0$ for all time.

On the other boundary $x = L$ (corresponding to the N'th mass),

$$m\frac{\mathrm{d}^2 u_N}{\mathrm{d}t^2} = k(u_N - u_{N-1}) \tag{1.33}$$

which in terms of the continuous variables becomes

$$\frac{\partial^2 u}{\partial t^2}(L,t) = \left(\frac{E}{\rho l}\right)\frac{\partial u}{\partial x}(L,t) \tag{1.34}$$

As we make $l \to 0$, the left hand side remains finite, but the right hand side would tend to infinity, unless

$$\frac{\partial u}{\partial x}(L,t) = 0, \tag{1.35}$$

for all t. We thus see that the free end of the elastic material is modelled by setting the derivative of u to zero.

1.3.2 Acoustic waves in a tube

Consider the propagation of sound waves in a fluid (a gas or liquid) along a long tube whose diameter is small, so that when sound propagates in the tube, all points on a plane at right angles to the axis move together. Let x be the position of a plane of particles before the sound wave is present. With the sound, let $u(x,t)$ denote the amount by which the plane which was at x is displaced at time t.

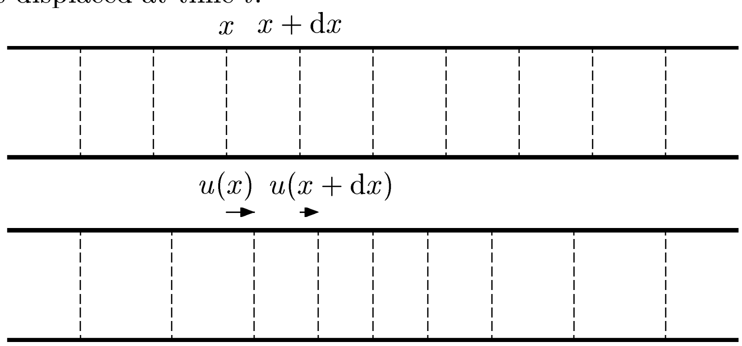

Figure 1.8: One dimensional sound propagation. Upper figure shows planes of fluid with no sound, lower figure shows planes of fluid with sound.

We first consider the change in volume of a portion of fluid which is originally between x and $x + \mathrm{d}x$, and with transverse area A. With the sound, the plane at x moves to $x + u(x,t)$ and the plane at $x + \mathrm{d}x$ moves to $x + \mathrm{d}x + u(x + \mathrm{d}x, t)$. The original volume is $A\mathrm{d}x$, and the volume with the sound is

$$A\left[\mathrm{d}x + u(x+\mathrm{d}x,t) - u(x,t)\right] \approx A\left[1 + \frac{\partial u}{\partial x}(x,t)\right]\mathrm{d}x \tag{1.36}$$

Suppose that the pressure in the fluid without sound is P_0. When the sound is present, the pressure is changed slightly to $P_0 + p(x,t)$. The *bulk modulus* K of the fluid is defined so that the pressure change p needed to cause a change of volume ΔV in a volume V of fluid is

$$p = -K\frac{\Delta V}{V}. \tag{1.37}$$

The negative sign is present to make K positive, since an increase in pressure causes a reduction of the volume. For the sound wave,

$$p(x,t) = -K\frac{A\left[1 + \frac{\partial u}{\partial x}(x,t)\right]\mathrm{d}x - A\mathrm{d}x}{A\mathrm{d}x} = -K\frac{\partial u}{\partial x}(x,t). \tag{1.38}$$

We now have the pressure at each point in the fluid. Next consider the longitudinal acceleration of an element of fluid between x and $x + \mathrm{d}x$. The force on this element is due to the difference in pressure between the two faces of the element, and the mass is given by $\rho A \mathrm{d}x$ where ρ is the (average) fluid density. Using Newton's second law,

$$\begin{aligned} \rho A \mathrm{d}x \frac{\partial^2 u}{\partial t^2} &= \left[p(x,t) - p(x+\mathrm{d}x,t)\right] A \\ &= \left[-K\frac{\partial u}{\partial x}(x,t) + K\frac{\partial u}{\partial x}(x+\mathrm{d}x,t)\right] A \\ &= K\frac{\partial^2 u}{\partial x^2}\mathrm{d}x\, A \end{aligned} \tag{1.39}$$

Hence,

$$\frac{\partial^2 u}{\partial t^2} = \frac{K}{\rho}\frac{\partial^2 u}{\partial x^2} \tag{1.40}$$

which is a one dimensional wave equation for propagation with speed $\sqrt{K/\rho}$.

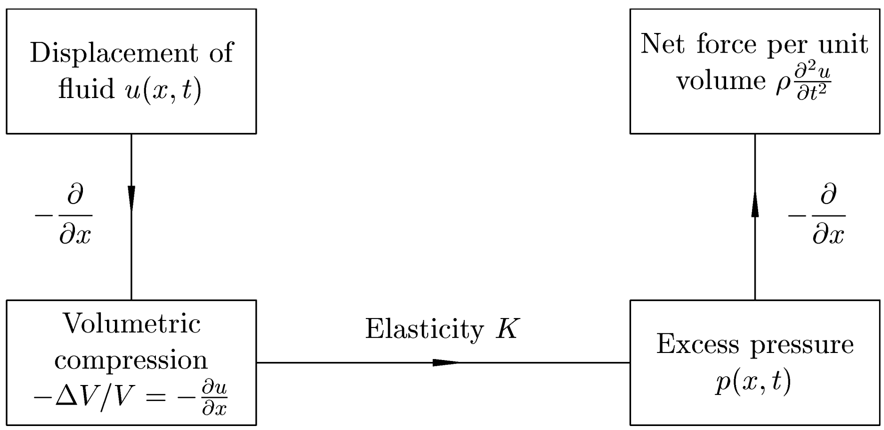

Figure 1.9: One dimensional sound propagation in general framework

1.4 Equilibrium of a system of elastic bars

We now consider a two-dimensional example in which a structure made up of elastic bars distorts under applied loads until it comes to equilibrium. For small displacements and extensions, elements such as beams of wood or steel used to make buildings and bridges can be approximated as elastic bars if the effects of bending are unimportant (Note: the usual terminology in engineering is that a *bar* resists stretching and compression, while a *beam* resists bending). Equilibrium studies of such structures allows us to work out how strong such members need to be in order to support the loads applied.

In Figure 1.10, the two lower suppports are assumed to be in fixed positions. The weights of the bars are assumed to be negligible, and the tension in each bar is assumed to be proportional to its extension. Prescribed forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the nodes 1 and 2 respectively and we seek to determine the (small) amount by which the structure distorts in order to come to equilibrium.

The unknowns are the displacements of nodes 1 and 2. Let us denote the displacements of the nodes from the equilibrium positions by $(\delta x_1, \delta y_1)$ and $(\delta x_2, \delta y_2)$ respectively. From the

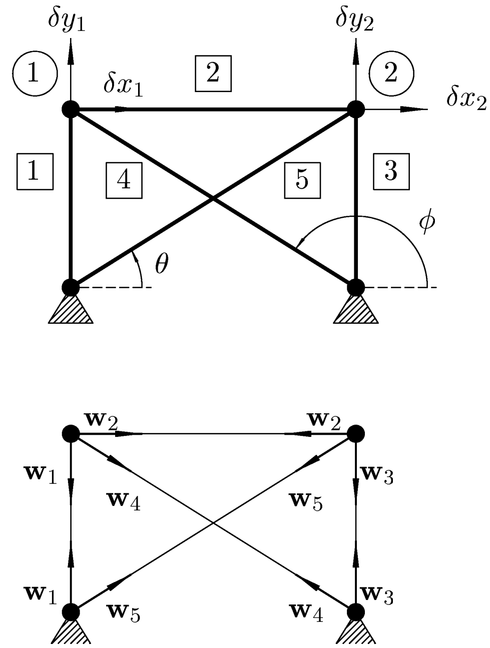

Figure 1.10: Equilibrium of structure consisting of elastic bars. Node labels are circled and bar numbers are in squares. Upper figure shows displacements and lower figure shows internal forces. External forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (not shown) are applied to the nodes.

nodal displacements, we need to find the extension of each bar. To do this, we first derive a general result for the change in length of a bar when its endpoints are moved slightly.

1.4.1 First-order change in length of a bar

Consider a bar whose endpoints are at (x_1, y_1) and (x_2, y_2). The length of the bar is then $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Now let us suppose that the endpoints are displaced by small amounts $(\delta x_1, \delta y_1)$ and $(\delta x_2, \delta y_2)$, as shown in Figure 1.11

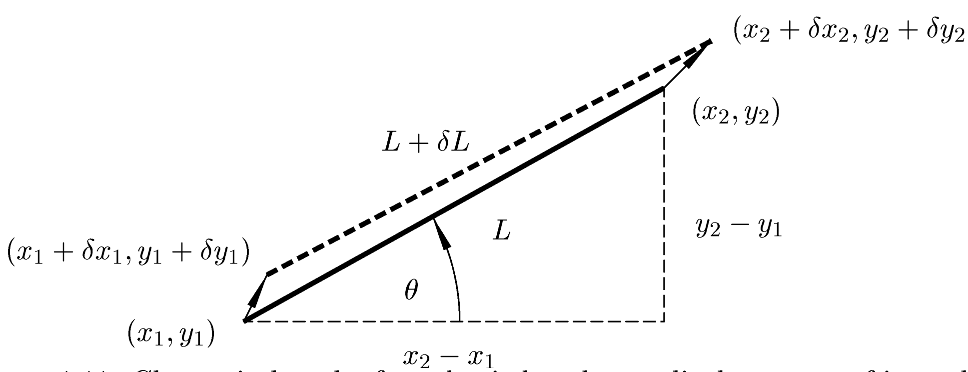

Figure 1.11: Change in length of an elastic bar due to displacements of its ends.

The new length of the bar is

$$\begin{aligned} L+\delta L &= \sqrt{(x_2-x_1+\delta x_2-\delta x_1)^2+(y_2-y_1+\delta y_2-\delta y_1)^2} \\ &\approx \sqrt{(x_2-x_1)^2+2(x_2-x_1)(\delta x_2-\delta x_1)+(y_2-y_1)^2+2(y_2-y_1)(\delta y_2-\delta y_1)} \\ &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\sqrt{1+\frac{2(x_2-x_1)(\delta x_2-\delta x_1)+2(y_2-y_1)(\delta y_2-\delta y_1)}{(x_2-x_1)^2+(y_2-y_1)^2}} \end{aligned}$$

where quadratic terms involving $(\delta x_2-\delta x_1)^2$ and $(\delta y_2-\delta y_1)^2$ have been neglected. Recall that the binomial approximation states that if $\varepsilon \ll 1$, $(1+\varepsilon)^n \approx 1+n\varepsilon+\ldots$. Applying this to the second square root, we see that

$$L+\delta L \approx \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\left[1+\frac{(x_2-x_1)(\delta x_2-\delta x_1)+(y_2-y_1)(\delta y_2-\delta y_1)}{(x_2-x_1)^2+(y_2-y_1)^2}+\ldots\right]$$

so that

$$\delta L \approx \frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(\delta x_2-\delta x_1)+\frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}(\delta y_2-\delta y_1).$$

In this approximation, the extension of the bar is a linear function of the displacements of the endpoints. Refering back to Figure 1.11, it is easy to see that the coefficients of the linear combination may be written as:

$$\delta L = \cos\theta\;(\delta x_2-\delta x_1)+\sin\theta\;(\delta y_2-\delta y_1) \tag{1.41}$$

where θ is the angle between the horizontal and the line joining (x_1,y_1) to (x_2,y_2).

We now return to the truss shown in Figure 1.10. Using the above result, it is straightforward to write the extensions of bars in terms of the displacements of the nodes in the original structure:

$$\underbrace{\begin{pmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5 \end{pmatrix}}_{\mathbf{e}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \cos\phi & \sin\phi & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \delta x_1\\ \delta y_1\\ \delta x_2\\ \delta y_2 \end{pmatrix}}_{\mathbf{u}} \tag{1.42}$$

The tension in each bar is related to the extension via Hooke's law. If the spring constants are denoted by k_i, and the tensions in the bars by $\mathbf{w}$,

$$\underbrace{\begin{pmatrix} w_1\\ w_2\\ w_3\\ w_4\\ w_5 \end{pmatrix}}_{\mathbf{w}} = \underbrace{\begin{pmatrix} k_1 & & & & \\ & k_2 & & & \\ & & k_3 & & \\ & & & k_4 & \\ & & & & k_5 \end{pmatrix}}_{\mathbf{C}} \underbrace{\begin{pmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5 \end{pmatrix}}_{\mathbf{e}} \tag{1.43}$$

Finally, consider the forces acting on the nodes. The force on the each node may be resolved into the x and y directions. The contribution from the internal (elastic) forces due to the bars is

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & -\cos\phi & 0 \\ -1 & 0 & 0 & -\sin\phi & 0 \\ 0 & -1 & 0 & 0 & -\cos\theta \\ 0 & 0 & -1 & 0 & -\sin\theta \end{pmatrix}}_{-\mathbf{A}^{\mathrm{t}}} \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}}_{\mathbf{w}}. \tag{1.44}$$

Make sure that you understand how this was written down. Consider for example the third row of the matrix which relates to the x component of the force on node 2. From figure 1.10, we see that tensions $\mathbf{w}_2$ and $\mathbf{w}_5$ have components in that direction. Since $\mathbf{w}_2$ acts along the $-x$ direction and the component of $\mathbf{w}_5$ along the x direction is $-w_5 \cos\theta$, we see that the total internal force on node 2 in the x direction is $-w_2 - \cos\theta\, w_5$ as given in the above matrix. Examining the resulting matrix shows that it is just $-\mathbf{A}^{\mathrm{t}}$, where $\mathbf{A}$ is the matrix relating $\mathbf{e}$ to $\mathbf{u}$.

At equilibrium, these internal forces must add to the external (applied) forces at the nodes to give zero. If we denote the applied forces by $\mathbf{f}$, with components f_{1x}, f_{1y}, f_{2x} and f_{2y}, the equilibrium condition is:

$$\underbrace{\begin{pmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 & 1 & 0 & -\cos\phi & 0 \\ -1 & 0 & 0 & -\sin\phi & 0 \\ 0 & -1 & 0 & 0 & -\cos\theta \\ 0 & 0 & -1 & 0 & -\sin\theta \end{pmatrix}}_{-\mathbf{A}^{\mathrm{t}}} \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}}_{\mathbf{w}} = \mathbf{0} \tag{1.45}$$

In summary, we have the relations

$$\mathbf{e} = \mathbf{A}\mathbf{u} \tag{1.46}$$

$$\mathbf{w} = \mathbf{C}\mathbf{e} \tag{1.47}$$

$$\mathbf{f} - \mathbf{A}^{\mathrm{t}}\mathbf{w} = \mathbf{0} \tag{1.48}$$

which may be combined to give

$$\mathbf{A}^{\mathrm{t}}\mathbf{C}\mathbf{A}\mathbf{u} = \mathbf{f} \tag{1.49}$$

Provided that $\mathbf{A}^{\mathrm{t}}\mathbf{C}\mathbf{A}$ is invertible, this allows us to find the displacements for a given loading. This problem again fits naturally into the general framework, even though the geometry as specified by the matrix $\mathbf{A}$ is more complicated than for the one dimensional examples we have encountered previously.

1.5 Steady-state current flow in a network

As a final example, consider solving for the voltages at the nodes of the resistive network shown in Figure 1.13. Node 0 is regarded as the reference node with respect to which voltages at the other nodes are measured. The basic physics involved is Ohm's law which relates the potential difference (voltage) across a resistor to the current flowing **through that resistor.** In order to solve the problem, we need to relate the potential differences to the nodal voltages, and satisfy Kirchoff's current law at each node.

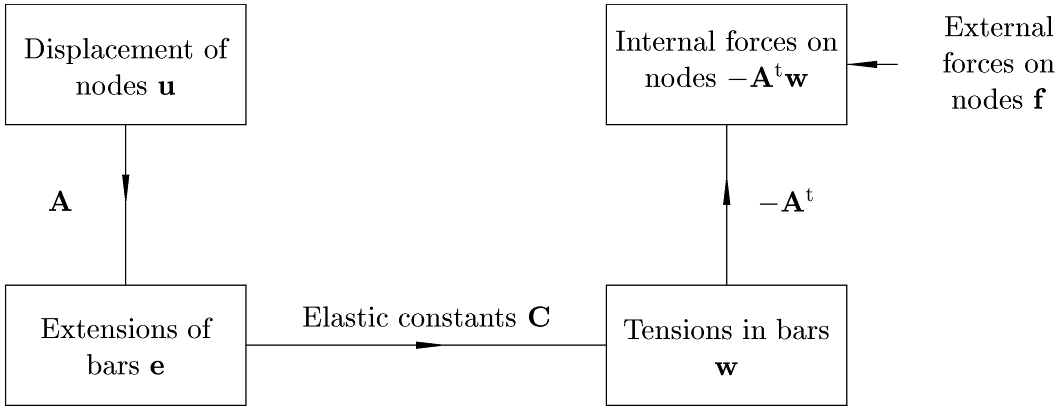

Figure 1.12: Equilibrium of structure consisting of elastic bars

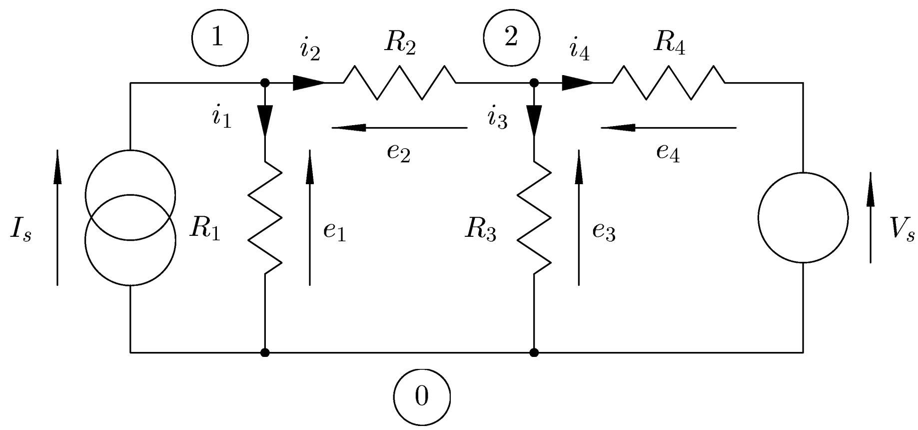

Figure 1.13: Voltages and currents in resistive network

Let the unknown voltages at nodes 1 and 2 be the components of the vector $\mathbf{V}$. The potential differences across the resistors, which are denoted in Figure 1.13 by e_1 through e_4 are related to these nodal voltages via

$$\underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}}_{\mathbf{e}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}}_{\mathbf{V}} - \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ V_s \end{pmatrix}}_{\mathbf{b}}. \tag{1.50}$$

The matrix $\mathbf{A}$ is determined by which nodes each resistor is connected between and the vector $\mathbf{b}$ contains any external voltage sources. Once the potential differences are known, Ohm's law allows us to compute the currents through the resistors

$$\underbrace{\begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}}_{\mathbf{i}} = \underbrace{\begin{pmatrix} \frac{1}{R_1} & & & \\ & \frac{1}{R_2} & & \\ & & \frac{1}{R_3} & \\ & & & \frac{1}{R_4} \end{pmatrix}}_{\mathbf{C}} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}}_{\mathbf{e}}. \tag{1.51}$$

Note that the directions of the currents through the resistors must be drawn in a way which is consistent with the directions of the potential differences as shown in the figure.

Finally, we must satisfy Kirchoff's current law, which states that the algebraic sum of the currents entering each node must vanish, since charge does not accumulate at any node. It is useful to separate the currents flowing into a node into those arising from currents in the resistors and those arising from external current sources. Considering first the currents **entering** each node due to the resistor currents, we have

$$\underbrace{\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}}_{-\mathbf{A}^{\mathrm{t}}} \underbrace{\begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}}_{\mathbf{i}} \tag{1.52}$$

Notice carefully how the elements in the matrix were written. For example, when considering node 2, current i_2 enters the node, while currents i_3 and i_4 leave the node, thus leading to the signs shown. Examining this matrix, we see that it is just $-\mathbf{A}^{\mathrm{t}}$.

When the external current sources are included, we satisfy Kirchoff's current law by insisting that the sum of any external currents entering each node and the internal currents (calculated using $-\mathbf{A}^{\mathrm{t}}\mathbf{i}$) must be zero. In the example, the external current sources may be included in the vector $\mathbf{f}$, where

$$\underbrace{\begin{pmatrix} I_s \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}}_{-\mathbf{A}^{\mathrm{t}}} \underbrace{\begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix}}_{\mathbf{i}} = \mathbf{0} \tag{1.53}$$

We may combine all these results to yield the system of equations

$$\mathbf{f} = \mathbf{A}^{\mathrm{t}}\mathbf{C}\left(\mathbf{AV} - \mathbf{b}\right) \tag{1.54}$$

which may be solved for $\mathbf{V}$, once $\mathbf{f}$, $\mathbf{b}$, $\mathbf{A}$ and $\mathbf{C}$ are known. This analysis again fits into the general framework as shown in Figure 1.14.

1.6 Summary of framework

Although the examples which we have considered have come from a variety of physical situations, they share several common features which cause them to fit into the general framework. The lower two boxes of each framework diagram are related by some local physical law which

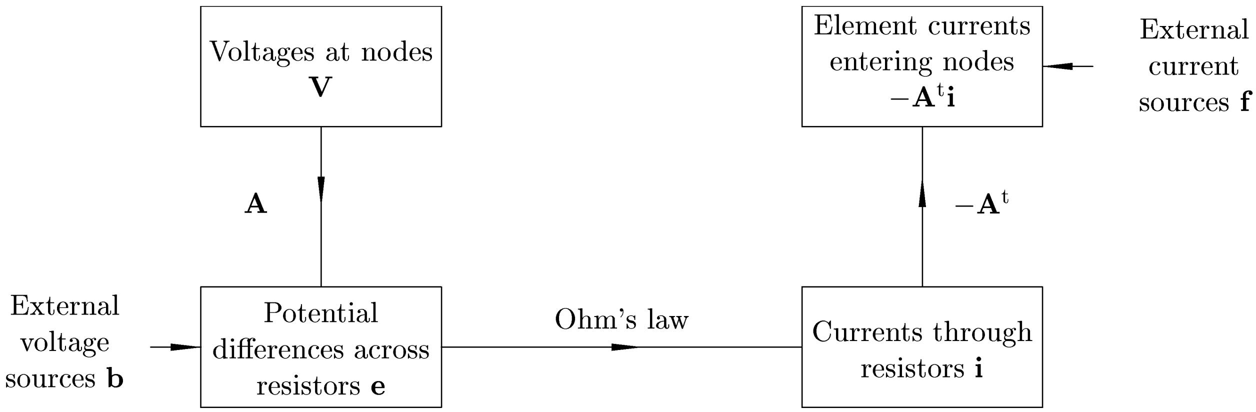

Figure 1.14: Steady-state current flow in general framework

applies to the elements which make up the system. This can be Ohm's law for resistors, Hookes' law for springs and elastic bars, the conduction law for the heat diffusion problem, etc. These relationships are local in the sense that they are statements about the elements concerned, and do not depend on how they are "hooked up" to form the system: for example, a given resistor will always obey Ohm's law, no matter what circuit it finds itself in. When this relationship is expressible as a matrix, the matrix involved is diagonal because of the locality.

The two vertical arms of the general framework diagram depend on the elements are connected together to form the system. For the one-dimensional continuous examples, each element is connected to its immediate neighbours on each side, and the operators involved are the spatial derivatives $\partial/\partial x$. From the example of masses and springs, we see that the derivatives may be thought of as a continuous limit of finite difference operators which arise because extensions are differences of displacements, and because the net force on each mass depends on the difference between the tensions in the springs to which it is attached. These lead to the matrices $\mathbf{A}$ and $-\mathbf{A}^{\mathrm{t}}$ in the two vertical arms in that example. If we look at the rows of $\mathbf{A}$ and $-\mathbf{A}^{\mathrm{t}}$, we see that both represent forward finite differences which in the continuous limit "become" $\partial/\partial x$.

In the last two examples involving structures and electronic circuits, the elements are interconnected in more complicated ways than in the one-dimensional examples. The general framework is nevertheless still valid, but with more complicated "geometry" matrices $\mathbf{A}$ and $-\mathbf{A}^{\mathrm{t}}$ becoming involved. Using vector calculus, two and three dimensional continuous systems can also be fitted into the framework. With a little more work, much of fluid dynamics and continuum mechanics (and electromagnetism, to some extent) can be be expressed in these terms.