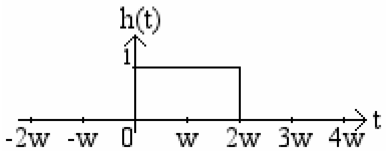
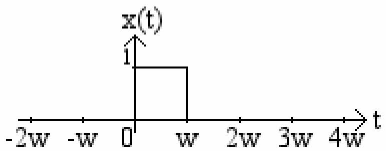
Convolution Example #1

Given:

$h(t) = P_{2w}(t - w)$



$x(t) = P_w(t - 0.5w)$



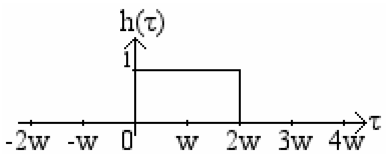
Where *w* is the width of the input rectangular pulse

Find:

$y(t) = x(t) * h(t) = h(t) * x(t) \; = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \; = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

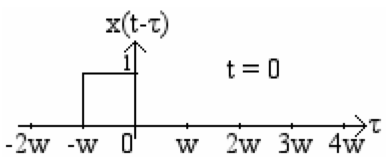
Where:

$$h(\tau) = P_{2w}(\tau - w) = \begin{cases} 0 & \tau < 0 \\ 1 & 0 \le \tau \le 2w \\ 0 & \tau > 2w \end{cases}$$

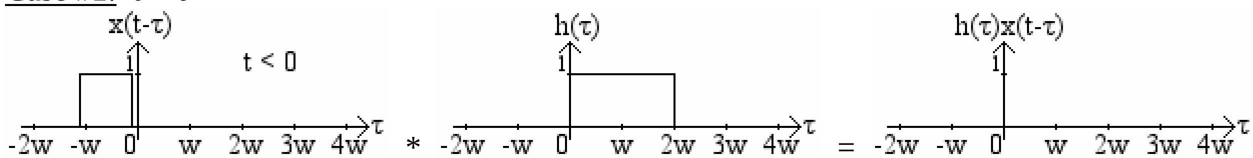


$$x(t-\tau) = P_w((t-\tau) - 0.5w) \; = \; P_w(t - \tau - 0.5w) \; = \; P_w(-(\tau - t + 0.5w)) \; = \; P_w\left(\frac{\tau - (t - 0.5w)}{-1}\right)$$

$$= x\left(\frac{\tau - t}{-1}\right) = \begin{cases} 0 & \tau < -w + t \\ 1 & -w + t \le \tau \le 0 + t \\ 0 & \tau > 0 + t \end{cases}$$



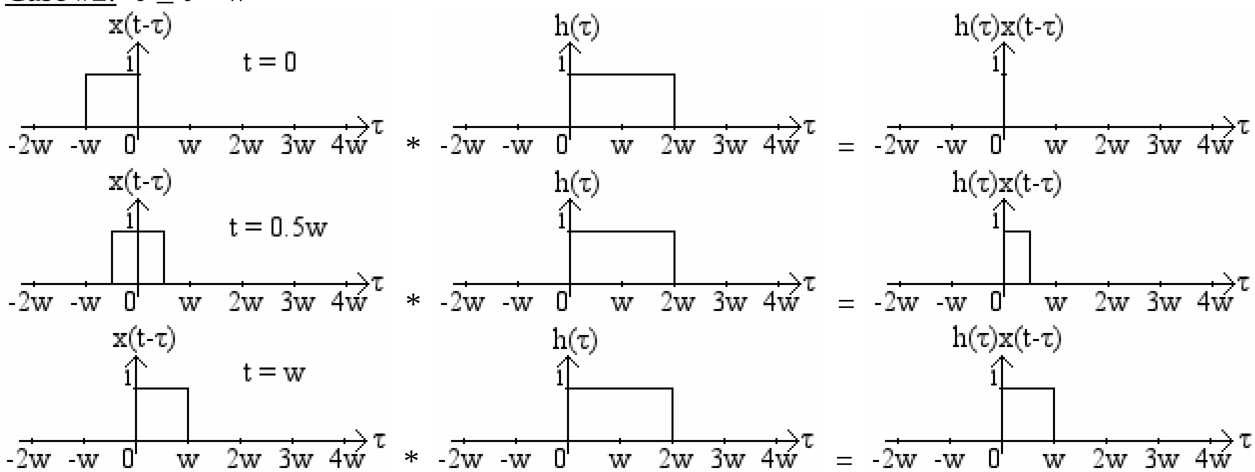
Case #1: $t < 0$



$$h(\tau)x(t-\tau) = 0$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

Case #2: $0 \le t < w$



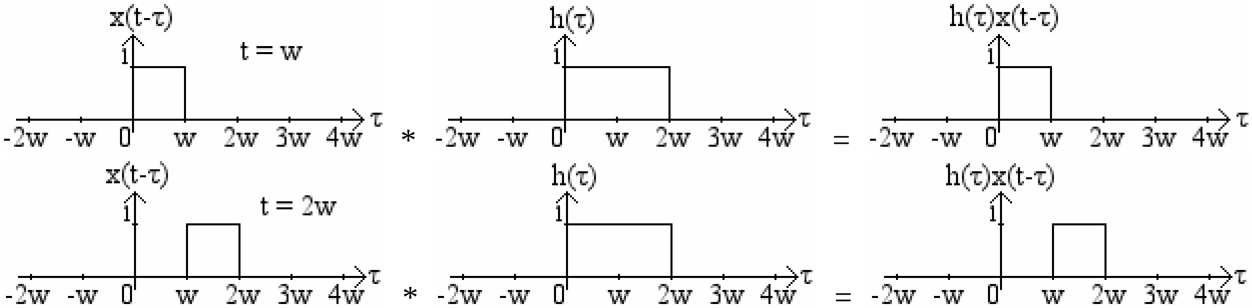
t	Lower Limit	Upper Limit
0	0	0
0.5w	0	0.5w
w	0	w

Lower limit of integration of h(τ)x(t-τ) as a function of t: $f_{Lower}(t) = 0$
Upper limit of integration of h(τ)x(t-τ) as a function of t: $f_{Upper}(t) = t$

$$h(\tau)x(t-\tau) = 1$$

$$y(t) = \int_0^t h(\tau)x(t-\tau)d\tau = \int_0^t 1\, d\tau = \tau\Big|_0^t = t - 0 = t$$

<u>Case #3:</u> w ≤ t < 2w



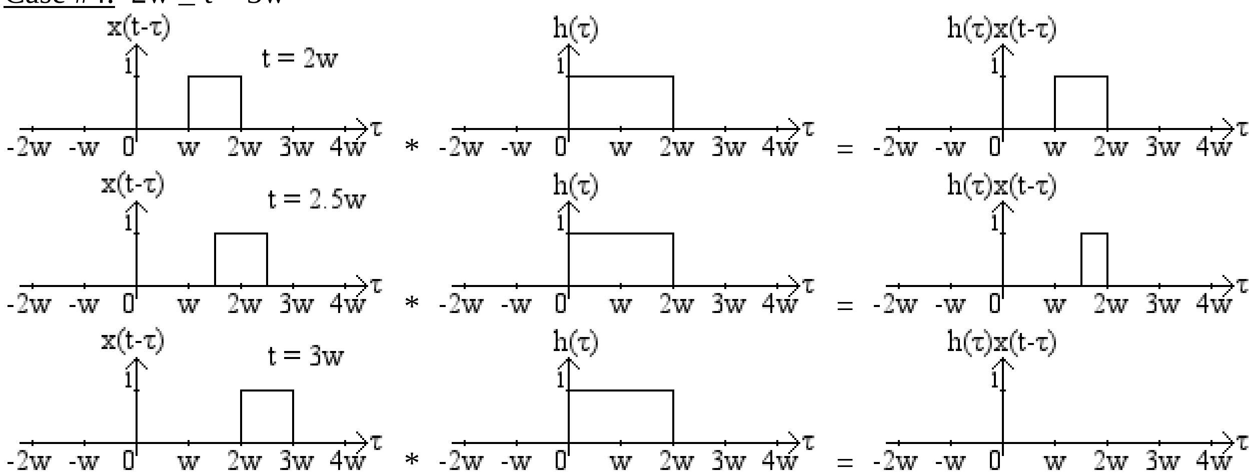
t	Lower Limit	Upper Limit
w	0	w
2w	w	2w

$f_{Lower}(t) = t - w$
$f_{Upper}(t) = t$

$$h(\tau)x(t-\tau) = 1$$

$$y(t) = \int_{t-w}^{t} h(\tau)x(t-\tau)d\tau = \int_{t-w}^{t} 1\,d\tau = \tau\Big|_{t-w}^{t} = t - (t-w) = w$$

<u>Case #4:</u> 2w ≤ t < 3w



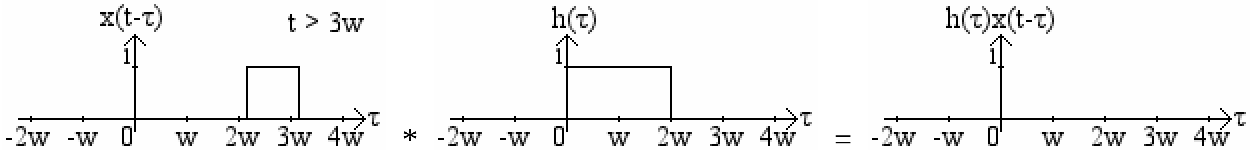
t	Lower Limit	Upper Limit
2w	w	2w
2.5w	1.5w	2w
3w	2w	2w

$f_{Lower}(t) = t - w$
$f_{Upper}(t) = 2w$

$$h(\tau)x(t-\tau) = 1$$

$$y(t) = \int_{t-w}^{2w} h(\tau)x(t-\tau)d\tau = \int_{t-w}^{2w} 1\,d\tau = \tau\Big|_{t-w}^{2w} = 2w - (t-w) = -t + 3w$$

Case #5: $t \geq 3w$



$$h(\tau)x(t-\tau) = 0$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} 0\, d\tau = 0$$

In Summary:

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < w \\ w & w \leq t < 2w \\ -t + 3w & 2w \leq t < 3w \\ 0 & t \geq 3w \end{cases}$$