A Constructive Approach to Constrained Hexahedral Mesh Generation

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ABSTRACT

S. Mitchell proved that a necessary and sufficient condition for the existence of a topological hexahedral mesh constrained to a quadrilateral mesh on the sphere is that the constraining quadrilateral mesh contains an even number of elements. S. Mitchell's proof depends on S. Smale's theorem on the regularity of curves on the compact manifolds.

Although the problem on the existence of constrained hexahedral meshes has been solved, the known solution is not easily programmable; indeed, there are cases, such as Schneider's pyramid, that are not easily solved.

A constructive proof to the existence theorem is provided in this paper for the sphere. The edges of the hexahedral elements that result from the construction are guaranteed to be linear. The construction generates *76*n* hexahedra elements where *n* is the number of quadrilaterals, is easily programmable, and solves open problems by R. Schneiners and D. Eppstein.

The results provided in this paper will be used in conjunction with S. Mitchell's Geode-Template to create an alternative way of creating a constrained hexahedral mesh.

Introduction

Hexahedral mesh generation has been subject to active research during the past twenty years. Progress has been made in the area of push-button hexahedral mesh generation, and tools exist that allow users to generate conformal hexahedral meshes for fairly specialized class of complex domains, but don't currently support all types of domains and applications.

The existence theorem proven by S. Mitchell [Mi-95] states that a solid homeomorphic to the sphere, whose boundary is tessellated by an even number of quadrilaterals, can be partitioned to a hexahedral mesh using interior surfaces whose boundaries are the dual cycles of the quad mesh. The solid partition is referred as the constrained hexahedral mesh, and the partition of the boundary is known as the constraining quadrilateral mesh.

The problem of constructing constrained hexahedral meshes has proven very difficult to address. The techniques based on S. Mitchell's proof to the existence theorem are difficult to implement; in a few cases, seemingly simple problems are difficult to solve. All implementations of algorithms related to S. Mitchell's proof build hexes with linear edges, but this does not guarantee that the elements generated have an acceptable quality.

D. Eppstein [Ep-96] presented a complexity analysis on the generation of constrained hexahedral meshes constrained to a bipartite quadrilateral mesh. Part of his construction depends on adding a layer of cells that have sixteen and eighteen faces; the problem of constructing the hexahedral solution to these cells of quadrilaterals is left open to the reader, and, instead, S. Mitchell's proof is invoked to prove existence of a solution to those cells. In his paper, D. Eppstein focuses on the analysis of the complexity of the generation of constrained hexahedral meshes.

In this paper, a constructive proof is given based on adding four basic canonical cells of hexahedral elements to a quadrilateral mesh: 1) a layer of paired hexahedra, 2) a layer of two split hexahedra, 3) a layer of elements to pair isolated quadrilaterals, and 4) a layer of four split pyramid elements. (The four types of transition elements are illustrated in Figure 14.) The rules of how to build layers of hexahedra using these canonical cells will be given.

The result, like S. Mitchell's construction, is topological. The one in this paper is a constructive easily programmable solution that does not depend on the existence theorem by S. Smale. A precise count on the number of hexahedral elements is provided.

S. Mitchell [Mi-98] introduced the Geode-Template to interface a Four Split quadrilateral mesh to a diced tetrahedral mesh. In his paper, Mitchell relies on splitting a hexahedral mesh into an eight-fold to create the Four Split or diced quadrilateral boundary. In this paper, it will be proven that it is not necessary to dice the hexahedral boundary; this will be an outcome of Theorem 5 below. The applications of this theorem to the Geode-Template will also be discussed.

The remainder of this paper will outline additional concepts, definitions, and proofs which ultimately result in a constructive proof of S. Mitchell's existence theorem. The proof of the theorem presented in this paper can be summarized by the following:

1. The notions of a Paired Partition and transition of quadrilateral meshes are introduced. It is proven that every quadrilateral mesh that admits a Paired Partition has a transition to a quadrilateral mesh whose dual has no self-intersecting loops.
2. The notion of four splits and the corresponding hexahedral solution to these special quadrilateral meshes is discussed. It is proven that a Four Split Mesh of a quadrilateral mesh of the sphere is the boundary a hexahedral mesh. The Four Split Pyramid cell is used to construct the solution.
3. It is proven that, if each of the dual loops of the quadrilaterals on the sphere does not self-intersect, there is a transition of the mesh to a Four Split Quadrilateral Mesh. The proof is done by inserting layers of elements that transition to sets of two quadrilaterals along the cycles.
4. A quadrilateral mesh on the sphere that admits a Paired Partition is proven to be the boundary of a hexahedral mesh of the interior of the sphere. A layer of hexahedra is added to transition to a quadrilateral mesh with no self-intersecting cycles.
5. It is proven that any quadrilateral mesh on the sphere with an even number of quadrilaterals can be transitioned into a quadrilateral meshes that admit a Paired Partition. With this result the proof is finished.

The mesh of hexahedra generated by the construction presented in this paper contains layers of elements that share a vertex at the center of the sphere. This mesh does not provide practical applications and is presented merely to provide a concrete measurable construction of a solution to the problem of constrained mesh generation.

Finally, a few solutions to open problems in mesh generation are presented including: new solution is given to Steiner's open problem, the eight sided quadrilateral octahedron [Ep-96], and Eppstein's cubes [Ep-96]. The solutions to these problems contain elements that share two faces. However, these irregularities can be removed by adding additional elements; the techniques needed to resolve these irregularities will be discussed at the end of the paper. A question by M. Bern and others [BEK+99] on the existence of a hexahedral decomposition with linear edges for a convex polyhedron is solved by the construction provided in this paper.

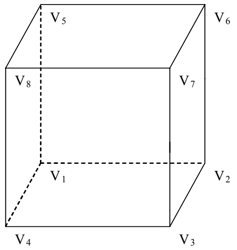
Basic Terminology

The terms quadrilateral and hexahedral mesh follow the definition given by S. Mitchell [Mi-95]. The dual of a quadrilateral mesh is a graph where every vertex is connected to four other vertices. A structure referred as the Spatial Twist Continuum or STC for short is associated to this graph. In this definition, the notion of chord is introduced. A chord is as a chain of quadrilaterals that is constructed by traversing the adjacent quadrilaterals through opposite edges. A loop is a closed chord. In particular, for a closed compact manifold, every chord belongs to a loop.

Loops may self-intersect. T. Suzuki et al [Su-2005] gave a detailed description of how to untangle self-intersecting loops to be able to create the interior surfaces to the STC that would generate the mesh. He applied his results to resolve R. Schneider's pyramid [Sch-www].

Element Representation

The quadrilateral and hexahedral elements to be referenced throughout the various proofs to be presented will follow the conventions used in Finite Element Analysis. A quadrilateral is represented by an ordered set of vertices $\{v_1, v_2, v_3, v_4\}$, and bounded by the four edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, and $\{v_4, v_1\}$. The edges in the quadrilateral that do not share vertices are called opposite edges of a quadrilateral. A hexahedral is represented by an ordered set of vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, and bounded by the six faces $\{v_1, v_4, v_3, v_2\}$, $\{v_5, v_6, v_7, v_8\}$, $\{v_5, v_6, v_2, v_1\}$, $\{v_8, v_7, v_3, v_4\}$, $\{v_6, v_2, v_3, v_7\}$, and $\{v_1, v_5, v_8, v_4\}$.



Some of the requirements that a quadrilateral mesh should meet are that edges contain exactly two points, and are shared by at most two quadrilaterals. Similarly, the faces of a hexahedron contain exactly four vertices and are shared by at most two hexahedra.

Additional requirements of a valid quadrilateral mesh are that each edge in the mesh must contain exactly two distinct vertices (nodes), and each interior edge must be shared by exactly two quadrilaterals. Similarly, the faces of a hexahedron must contain exactly four distinct vertices (nodes), and each interior face of the hexahedral mesh must be shared by exactly two hexahedra.

Hexahedral Transitions of Quadrilateral Meshes

Definition 1 *Two distinct quadrilateral meshes are transitions of each other if there is a hexahedral mesh whose boundary is the union of both meshes.*

By solving the hexahedral mesh of the transition of a given quadrilateral mesh, the original hexahedral problem is resolved, because the union of the hexahedral mesh with the layer of transition elements gives the solution to the original quadrilateral mesh.

Paired Partition of a Quadrilateral Mesh

[Definition 2] *A* ***Paired Partition*** *of a quadrilateral mesh Q is a partition P_Q of Q such that each element in the partition is a pair of quadrilaterals that share at least an edge.*

In other words, a quadrilateral mesh Q admits a paired partition if there exist a set

1. $P_Q = \{ \{p, q\}$, such that p and q are in $Q\}$,
2. any two distinct elements in P_Q $\{p, q\}$ and $\{p', q'\}$ are disjoint,
3. Q is the union of P_Q , and,
4. for each element $\{p, q\}$ in P_Q, p and q share an edge or, equivalently, p and q are neighbors.

Removal of Self-intersecting Loops

[Th 1] *Every quadrilateral mesh on the sphere with n elements that admits a* ***Paired Partition*** *transitions to quadrilateral mesh with no self-intersecting loops. The total number of elements within the transition between the original mesh and the mesh with no self-intersecting loops is n.*

Proof:

Construct the ***Paired Partition*** of the quadrilateral mesh. Let {*p, q*} be in P_Q.

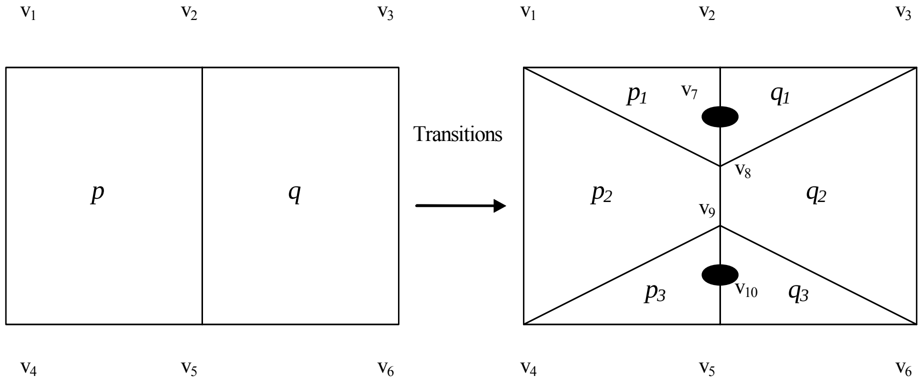


Figure 1

The two quadrilaterals with vertices {{v_1, v_2, v_5, v_4}, {v_2, v_3, v_6, v_5}} transition to the quadrilateral mesh with six quadrilaterals {{v_1, v_2, v_7, v_8}, {v_7, v_2, v_3, v_8}, {v_1, v_8, v_9, v_4}, {v_8, v_3, v_6, v_9}, {v_4, v_9, v_{10}, v_5}, {v_9, v_6, v_5, v_{10}}}. The boundary of the hexahedral mesh comprised of the two hexahedra with vertices {v_1, v_2, v_5, v_4, v_8, v_7, v_{10}, v_9}, and {v_2, v_3, v_6, v_5, v_7, v_8, v_9, v_{10}} mesh below is the union of the two sets of quadrilaterals. For any two paired quadrilaterals q_1, and q_2 with vertices {v_1, v_2, v_3, v_4}, and {v_2, v_5, v_6, v_3}, construct the two hexahedral elements with vertices {v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}}, and {v_2, v_5, v_6, v_3, v_8, v_7, v_{10}, v_9}.

The transition discussed above is applied to each set of ***Paired Quadrilaterals*** in the partition. The boundary of transition mesh minus the original quadrilateral mesh is composed by cells of six quadrilateral elements. Thus each paired element in the ***Paired Partition*** is mapped to a unique set of quadrilaterals {p_1, p_2, p_3, q_1, q_2, q_3} in the transitioned mesh.

There is a ***natural partition*** induced by the mapping each element {*p, q*} the ***Paired Partition*** to a unique subset of quadrilaterals {p_1, p_2, p_3, q_1, q_2, q_3} of the transitioned mesh; the new mesh can be partitioned into ***Cells of six quadrilaterals***. As a consequence of a peculiar property of these new cells, it will be proven that all the loops in the new mesh that results from the transition are non-self-intersecting as will be shown next.

For any quadrilateral in the transitioned quadrilateral mesh, find the element of six quadrilaterals in the ***natural partition*** that contains the quadrilateral. There are two types of loops that go through a ***Cell of six quadrilaterals***: there is one loop that is fully contained inside a ***Cell of six quadrilaterals***, and others that are not as Figure 2 illustrates. The only intersections in the cell containing this quadrilateral take place between the fully contained loop in the cell and a loop that is not fully contained in the cell; thus for the given quadrilateral the intersection take place between two distinct loops. Hence, there are not self-intersections.

A total of *n* hexahedral elements were used to transition to a quadrilateral mesh with no self-intersecting loops. It is possible that only a subset of the pairs is required to remove the singularities; that is, *n* is an upper bound on the minimum transition layer. This concludes the proof.

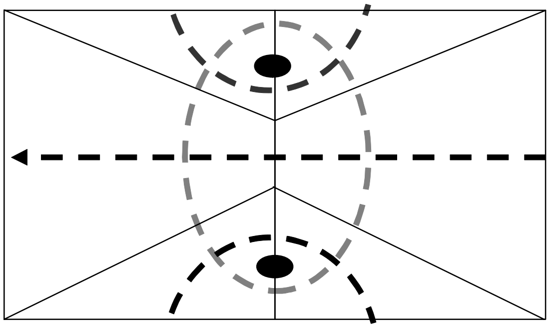


Figure 2

Four Splits

In this section, two types of transitions are discussed: a transition to a Four Split cell, and a Four Split Pyramid transition. The transitions to Four Split cells allow us to transition from Paired Partitions to Four Split Quadrilateral Meshes, and the Four-Split pyramid will be used to transition from cells of four quadrilaterals to an all-hexahedral mesh. Before continuing the discussion, the notion a Four Split Cell is introduced.

A **Four Split Cell** is a collection of four cells that result from splitting a quadrilateral into four quadrilaterals; five points are added: four at the mid-edges, and one at the center as shown in Figure 3.

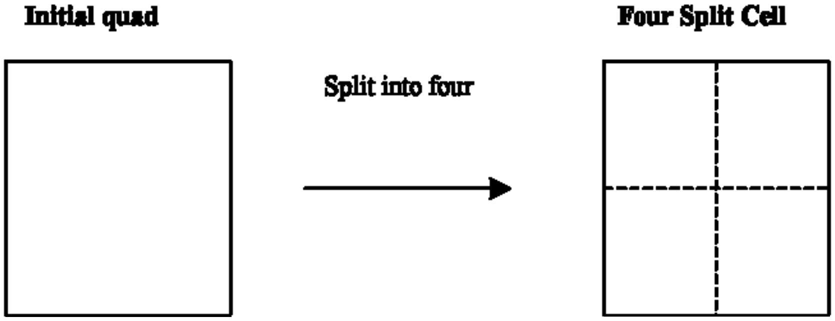


Figure 3

Any quadrilateral mesh that results from such a split is called a *Four Split Quadrilateral Mesh*.

A *Four Split Quadrilateral Mesh* has the following properties:

1. There are four vertices labeled as corner vertices.
2. There is a unique vertex labeled as interior.
3. There are four vertices labeled as mid-edges.

4. Adjacent cells share corner vertices with corner vertices, and mid-edge vertices with mid-edge vertices. This property is essential to the proof of the next theorem to ensure that the faces of the Four Split Pyramid given below can match properly.

Theorem 2 provides a sufficient condition for the existence of a transition to a ***Four Split Mesh***.

[*Th 2*] *If each of the dual loops of the quadrilaterals on the sphere does not self-intersect, there is a transition of the mesh to a four split quadrilateral mesh.*
Proof:

A given oriented loop *l* in the dual of the quadrilateral mesh splits the mesh into three disjoint sets. One corresponds to the sets of elements that lie to the left of the loop, and the other to the set of elements to the right, because there are no self-intersections that change the orientation of the curve, and the third to the elements in the loop.

There are three cases to consider when adding a layer: the first case is when a quadrilateral lies to the regions labeled as left of the oriented loop, the second case is when a quadrilateral lies to the right side of the loop, and the third case is when a quadrilateral is part of the loop. Each case will be discussed separately.

The first two cases are the simplest ones. If a quadrilateral lies to the right of an oriented loop, a single hexahedron is added on top towards the center of the sphere. In this case, there will be exactly one node on top of each vertex of the set of quadrilaterals.

When the quadrilateral is in the region labeled as left of the loop, two hexahedra are added on top of the quadrilateral towards the center of the sphere. On top of each vertex of the quadrilaterals there will be two vertices.

The remaining case is the case of a quadrilateral that belongs to the loop. For this case, there are three possible scenarios: in the first scenario, there is quadrilateral to the left or right that is not part of the loop, in the second case there is a quadrilateral that belong also to the loop to either of its sides. Each case will be discussed separately.

Create the following transition hexahedral elements on each quadrilateral of the mesh.

1. If a quadrilateral element is to the right of the loop, add a hexahedron to the quadrilateral towards the interior of the sphere.
2. If the quadrilateral is to the left of the loop, add two hexahedra on top of each other.
3. If the quadrilateral corresponds to the loop, add the following transition element illustrated in Figure 4.

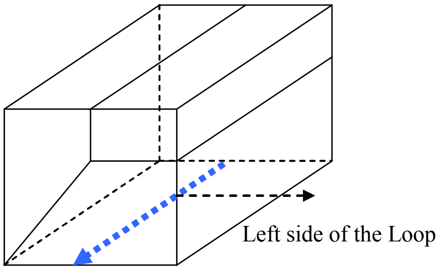


Figure 4

The combination of the three follows:

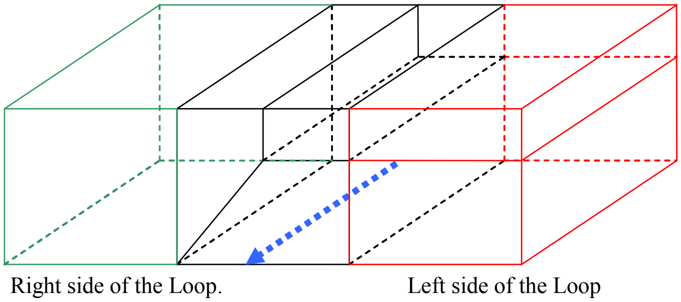
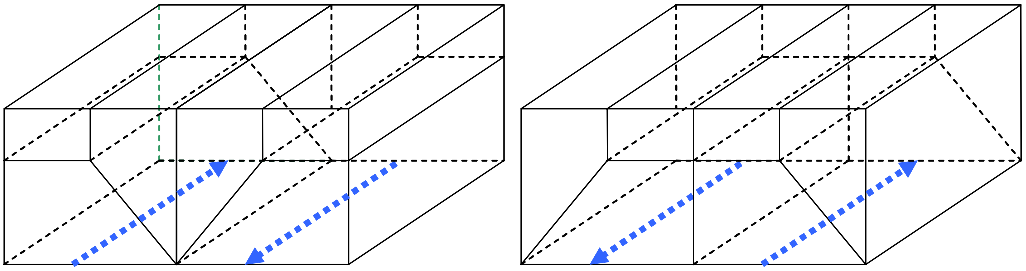


Figure 5

The second case is where two adjacent quadrilaterals belong to the same cycle. In this case, the loops meet at opposite directions; the reason is given next. Construct the curve composed by any of the two edges of the adjacent quadrilateral not parallel to the loop. The intersection of the curve with the loop must do so at opposing directions; this is a direct result on the property of simple Jordan curves. (Look at the appendix titled Simple Jordan Curves for more details.) The two possible situations are when the loops meet either to the right of each other, or when they meet to the left of each other.



The loops are to the right of each other.

The loops are to the left of each other.

Figure 6

The total number of transition elements added is given by *number of transition elements = number of quadrilaterals to the right of the loop + 2*number of quadrilaterals to the left + 3*number of quadrilaterals from the loop*

The hexahedral elements transition the original quadrilateral elements to one where the quadrilaterals to the left and right of the mesh remain identical to the originals, but the ones along the loop are split into two quadrilaterals. The set of originals chords is transferred to the transition quadrilateral mesh as follows:

1. The loop *l* just processed is discarded from the set of chords,
2. The remaining chords are mapped onto the faces of the transition quadrilateral mesh. In particular, when a loop is projected onto one of the pair of quadrilaterals that result from the split along the loop *l*, it has to be transverse to the loop *l*.

The operation of adding two split transition layers described above is repeated for another loop l' of the remaining loops projected onto the new quadrilateral mesh that results from the transition elements. Each time a loop is process the set of remaining loops diminishes by one. The process continues until no more loops remain from the original set. Wherever two loops intersect, a group of **Four Split Cell** is created, because two quadrilaterals were result from the loop processed at an earlier stage in the process, and the newly processed loop adds two more quadrilaterals along the transversal direction.

Figure 7 below illustrates the results of two layers from two different loops that intersect.

At their intersection, the two layers of hexahedral transition elements create a four split.

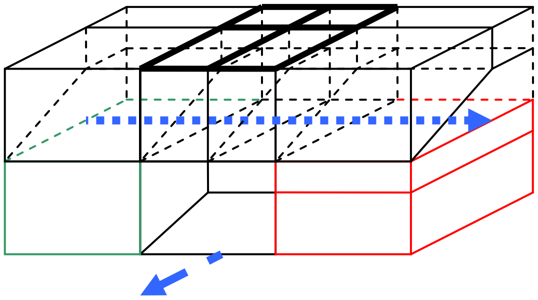


Figure 7

Each loop is processed only once, and exactly two loops cross each quadrilateral cell in traversal directions; hence, the resulting transition of quadrilaterals will be a **Four Split Mesh**.

It is critical for this construction the fact that the quadrilateral mesh is on the sphere. A torus, for example, has loops that do not split the domain in two regions as required by the proof. In general, non-intersecting loops could be processed simultaneously by the approach given above if the chords are carefully oriented reducing the number of layers.

Once a Four Split Quadrilateral Mesh is created, it is possible to transition to a constrained hexahedral mesh. The proof to the next theorem explains how this is done.

[Th 3] *A* ***Four Split Quadrilateral Mesh*** *of a quadrilateral mesh of the sphere is the boundary a hexahedral mesh containing 16*<number of quadrilaterals before four-split division> hexahedra.*

Proof,

The construction is done by placing the base of the pyramid of hexahedra illustrated in Figure 8 at each of the ***Four Split Cells***. The pyramid contains sixteen hexahedra. (A detailed construction of the pyramid is given in [Ca-www].) Some hexahedra share two faces with one of their neighbors. The base of the pyramid is a four split cell.

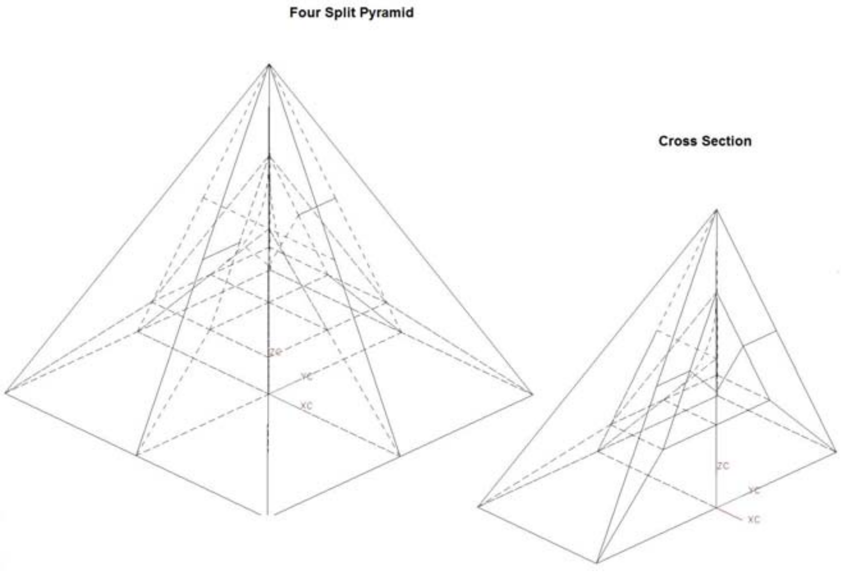


Figure 8

The hexahedral pyramids are placed inside the sphere with their bases aligned at each four split cell as illustrated below.

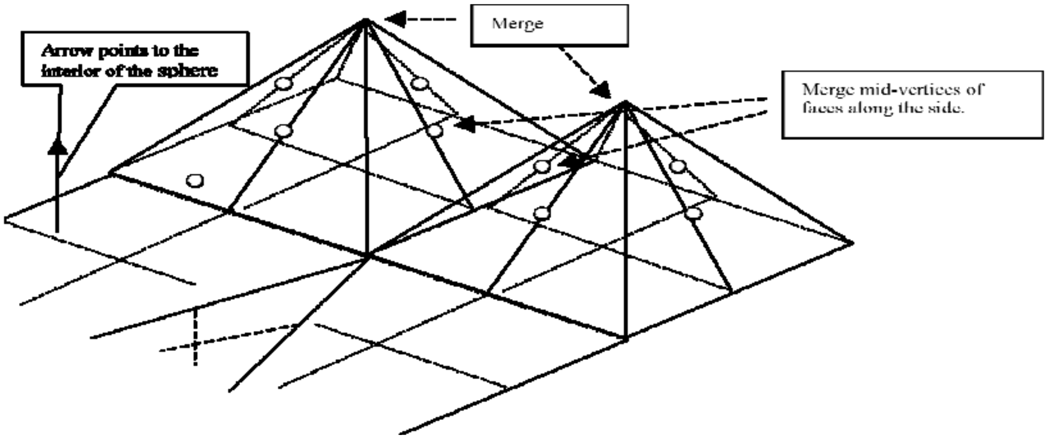


Figure 9

The apex of pyramids is connected to the center of the sphere, and the midpoints to the corresponding midpoints of the faces to the adjacent cells. This connectivity is guaranteed by the fundamental property of the *Four Split Quadrilateral Mesh* that ensures that corner vertices meet with comer vertices, and mid-edge vertices meet with mid-edge vertices.

Main Result

[Th 4] *A quadrilateral mesh on the sphere that admits a* ***Paired Partition*** *is the boundary of a hexahedral mesh of the interior of the sphere. The total number of hexahedra is* ***76*n;*** *where* ***n*** *is the number of quadrilaterals.*

Proof:

By Theorem 1, a transition to a quadrilateral mesh can be constructed where no loop self-intersects. By Theorem 2, the quadrilateral mesh transitions to a four split, and, by Theorem 3, the quadrilateral mesh is the boundary of a hexahedral mesh.

The total number of hexahedral elements is given by

*number of hexahedra = number of elements to resolve self-intersecting loops + number of elements to transition to a four split + number of four split pyramids * 16.*

1. The number of hexahedra added by applying the transition that removes self-intersecting loops illustrated in **Figure 1** is

   $n.$

2. For each of the six quadrilaterals of each cell, the total number of hexahedra needed to transition to a four split is 9; hence the total number of hexahedra needed to transition the mesh to a four split is

   $9 * 6 * n/2.$

Each cell, as **Figure 2** shows, contains two types of loops, the inner loops, and the exterior loops. None of the inner loops intersect each other; hence, it is possible to orient all the loops in a manner that their direction is consistent with the two coloring map result mentioned in the appendix on Jordan curves. Similarly, none of the exterior loops intersect each other, and hence the same orientation principle applies. Thus, it is possible, by adding two layers of two splits, to transition to a four split. Indeed, each cell is homeomorphic in the sense that the inner loops can be oriented so that the left side of the inner loop is pointing to itself. Each of the cells has two outer loops oriented to the left of each other, and the remaining loop to the right of the center loop. This limits what can happen to each cell as the layers are added to one scenario.

3. The total number of hexahedral elements per cell needed to solve the four split is

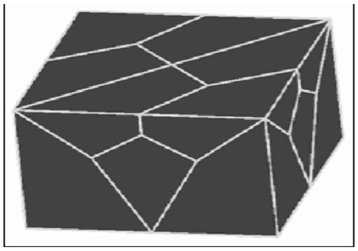
   $16 * 6 * n/2.$

*Hence, the total number of hexahedra to fill the interior is n + 48*n + 27*n which equals 76*n.*

> There are further constructions that could be made to remove the degeneracy of every element, and will increase considerably the number of elements. That is, it is possible to remove the fact that many elements share two faces or edges with neighboring elements.

S. Mitchell's Geode Template

A very attractive alternative to the Four Split Pyramid is S. Mitchell's Geode-Template.



The Geode-Template contains 26 hexahedral elements. This template is used to match a four split quadrilateral cell to a diced tetrahedral constrained mesh. The interior of the mesh is filled by two four element hexahedral dicing of a tetrahedral element where the apex is connected at the center of the sphere.

Thus, the solution to the constrained hexahedral mesh requires **4*2*6*n/2** elements coming from dicing the tetrahedral elements that connect their apex to the center of the sphere. In addition, there are **n + 9*6*n/2** hexahedra needed to transition to the Four Split Quadrilateral mesh and **26*6*n/2** elements from the Geode-Template. The new solution of the constrained hexahedral mesh contains a total of **130*n** elements.

One major benefit of using the Geode-Template is that the mesh will contain significantly less doublets than the one produced by the Four Split Pyramid.

Transition to Paired Partitions

It is clear that a necessary condition for a mesh to have a paired transition is that it has an even number of quadrilaterals. Although it will be established that any quadrilateral mesh on the sphere admits a transition to a mesh that admits a ***Paired Partition***, the question of whether or not every quadrilateral mesh on the sphere with an even number of elements admits a ***Paired Partition*** is open.

The following theorem is proven before the proving S. Mitchell's existence theorem:

[Th 5] *A quadrilateral mesh on the sphere with an even number of elements can transition to a quadrilateral mesh that admits a **Paired Partition**.*
Proof:

> Let Q be a quadrilateral mesh on the sphere. Let B be the set of all connected subsets of Q that admit ***Paired Partition.*** This set is not empty because any quadrilateral q in Q may be paired with an adjacent quadrilateral q'. Thus the set $\{q, q'\}$ belongs to B. The set B is a partially ordered set under the set containment relation. Thus, there is

an element *A* in B that is maximal. If *A* is *Q*, then *Q* would belong to *B* and would admit a ***Paired Partition***.

Suppose *A* is not *Q*. Any quadrilateral element *q* that does not belong to any pair in *A* must be surrounded by paired elements from *A*. If this were not the case, there would be a neighbor *n* of *q* such that for all *x* in *A*, the neighbor *n* is not in *x*. Form the pair $\{q, n\}$; the set *A* U $\{q, n\}$ formally contains *A*. This would violate the maximal property of *A*.

Thus, it is possible to split the set Q into paired quadrilaterals, and *isolated* quadrilaterals. That is

$$Q = (\text{Union of all pairs in } A) \cup \{q_1\} \cup \{q_2\} \cup \dots \cup \{q_k\}$$

The set of *isolated* quadrilaterals $\{q_1, q_2 \dots q_k\}$ must have an even number of quadrilaterals because the set *Q* contains an even and the union of all pairs in *A* contains an even number; that is

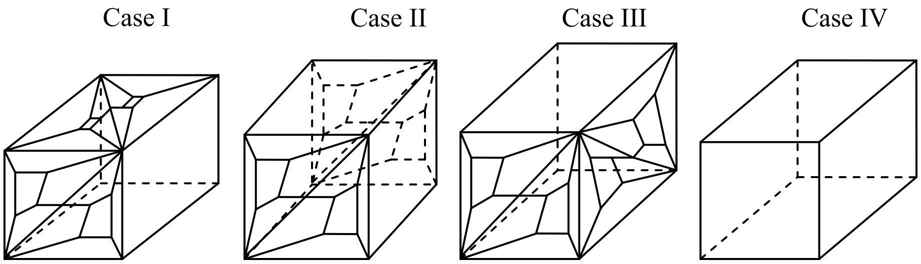
$$k = (\text{Cardinality of } Q) - (\text{Cardinality of the union of all pairs in } A).$$

Thus, ***k*** is an even number**.** (In the remaining of the proof, these **k** quadrilaterals are referred as isolated quadrilaterals.)

The dual of the quadrilateral mesh is a graph where each vertex or element in the mesh is connected to four vertices or neighbors as mentioned in section 1. Any two quadrilaterals are connected by a minimal path in a connected graph. Hence, for any two *isolated* quadrilaterals q_1 and q_2 there is a minimal path that connects them. The minimal property of the path prevents q_1 and q_2 to be in the path more than once; hence, they are strictly speaking endpoints to the path.

A construction used to transition to a quadrilateral mesh with fewer *isolated* quadrilaterals is given next. Add a layer of quadrilaterals to each quadrilateral that does not belong to the path connecting q_1 to q_2. At q_1 and q_2 add a layer of quadrilaterals with the following shape: There are ten quadrilaterals at the top of the layer, ten more at the face that is shared by the next element at the path, and one face at the three other sides, and one at the bottom. At each element in the path, add a layer of quadrilaterals such that the top is a single quadrilateral, and to the two sides connecting to the other elements in path we add the ten quadrilaterals.

Figure 10 shows four types of cells to be used in this construction. Case I is used at the endpoints of the path or, equivalently, on top of the isolated quadrilaterals, Case II is used when the path at a quadrilateral is moving from one edge to its opposite, and Case III is used when the path moves from one edge to an adjacent edge, Case IV is a hexahedra sitting on top of a quadrilateral that is not contained in the path. Case II admits a trivial hexahedral mesh defined by connecting vertices from one face to its opposite face. The cells in cases I and III admit a ***Paired Partition***, and can be solved by Theorem 4; the details of the paired partition are given in the section on applications.



Canonical Elements

Figure 10

The quadrilateral mesh that results from the transition is identical to the original quadrilateral mesh except for the ten quadrilaterals added at each of the isolated quadrilaterals connected by the path. Every other quadrilateral was not altered by this operation. The ten quadrilaterals can be paired independent from the rest of the mesh. Thus the new quadrilateral mesh that results from this transition has two less isolated quadrilaterals. This process is applied for each of the remaining quadrilaterals until none are left, and the resulting transition quadrilateral mesh has a paired transition that is similar to the original except that every isolated quadrilateral has been independently replaced by ten quadrilaterals. The set of pairs of quadrilaterals are added to the paired partition inherited from the original mesh, and, thus, giving the desired paired partition to the final transition. This finishes the proof.

S. Mitchell’s Existence Theorem

In this section we apply the previous results to provide an alternative proof to S. Mitchell’s existence theorem.

[Theorem] *A quadrilateral tessellation of the unit sphere with an even number of elements is the boundary of a hexahedral mesh of the unit ball.*

Proof,

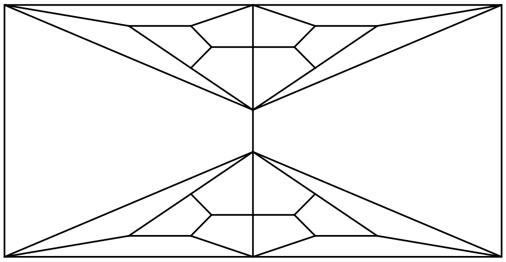
Any quadrilateral mesh on the sphere with an even number of quadrilaterals can be transitioned into a quadrilateral mesh on the sphere that admits a paired partition by Theorem 5. By Theorem 4, the union of the solution to the paired partition with the transition elements gives the desired hexahedral mesh.

Removal of Topological Irregularities

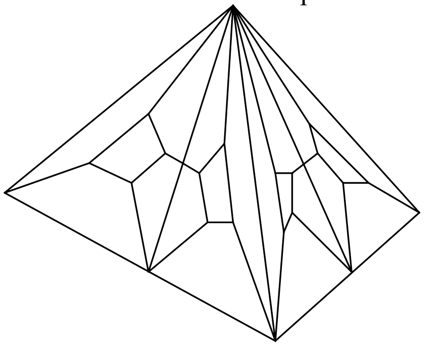
The mesh resulting from the construction contains many doublets or hexahedra that share two faces with the same neighbor. [Mi+94] S. Mitchell, T. Tautges give a detail description on how to remove doublets from a mesh while preserving the boundary.

A slightly different approach would replace the transition from the paired partition and the Four Split Pyramid by transition cells with no singularities by applying the pillowing of doublets as described by S. Mitchell and T. Tautges. The base of the new Four Split Pyramid will still be a four split quadrilateral cell.

Transition Cell for the Paired partition with no doublets



Isometric view of the Four Split Pyramid with no doublets. The base is still a four split cell



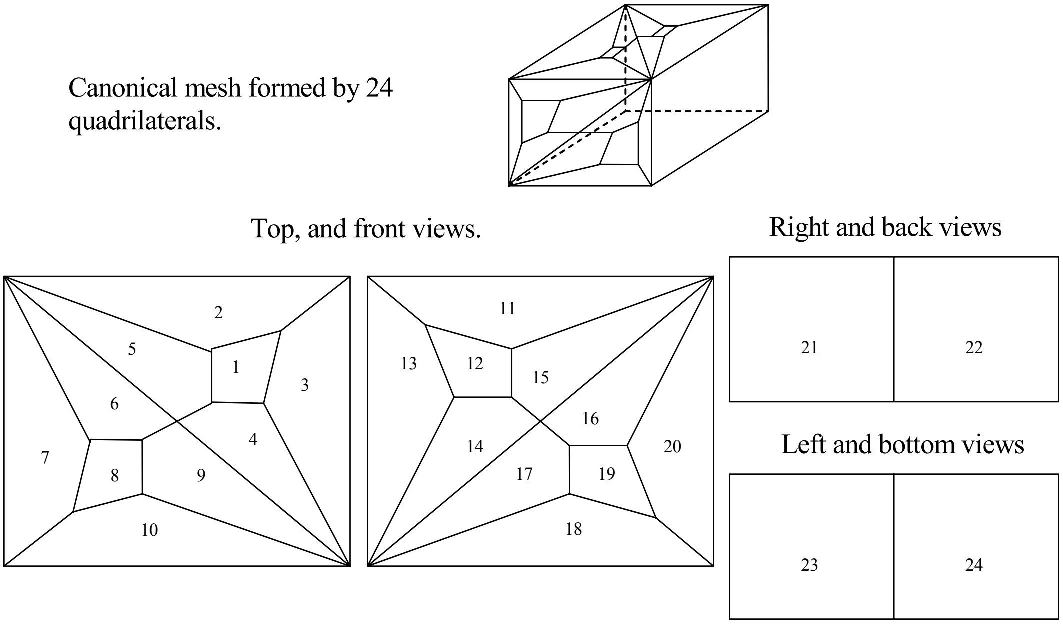
There are no self-intersecting loops in the transition cell without singularities.

Applications

The solutions to the quadrilateral meshes for Figure 10, Schneider's Pyramid, and D. Eppstein's Quadrilateral Octahedron are presented. In all cases, the solution is a direct result from Theorem 4. The solutions are presented by conveniently numbering the quadrilaterals such that the quadrilateral admits a ***Paired Partition*** of the form $P_Q = \{\{q_1, q_2\},\ldots,\{q_{2k-1}, q_{2k}\},\ldots,\{q_{2n-1}, q_{2n}\}\}$ where $2n$ is the number of quadrilaterals for each case.

Elements from Figure 10

Case II has a trivial hexahedral solution. Cases I and III, being topologically equivalent, admit an equivalent topological hexahedral solution. A total of 1824 hexahedral elements solve the interior by Theorem 4.


Canonical mesh formed by 24 quadrilaterals.

Top, and front views.

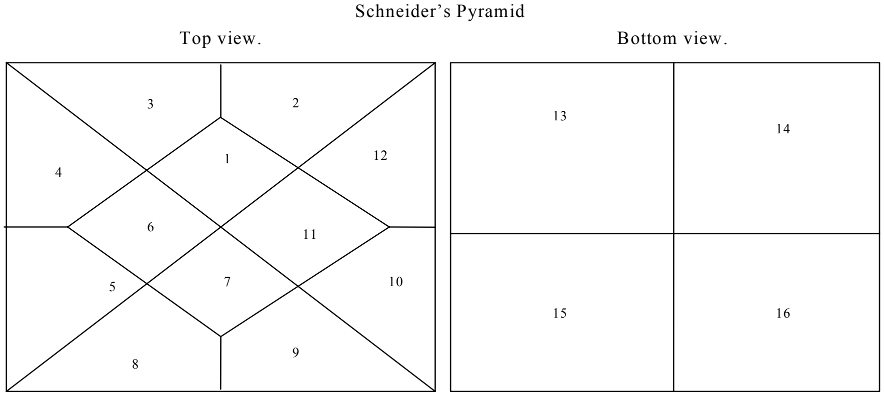
Right and back views

Left and bottom views

Figure 11

Schneider's Pyramid

Schneider's pyramid can be solved by directly applying Theorem 4. The figure below contains an open view of the pyramid and the numbered quadrilaterals.


Schneider's Pyramid

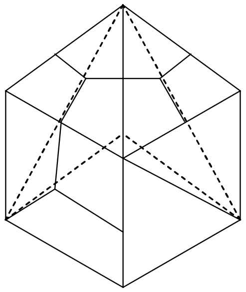
Top view.

Bottom view.

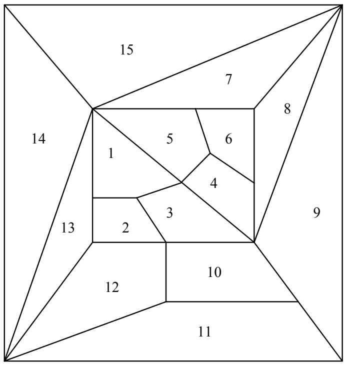
Eppstein's cubes

In Eppstein's original construction, there are two sets of quadrilateral cubes used to transition to the tetrahedral based mesh. One had sixteen elements, and the other eighteen. His original construction contained quadrilaterals that shared two sides with the same neighbor that is a highly undesirable condition, but, on the other hand this situation should not be of concern because it is fixable as proven in the section on quality improvements. There is another version of Eppstein's cubes with 22, and 20 quadrilateral elements respectively and none are degenerate; the solution is similar to the one below.

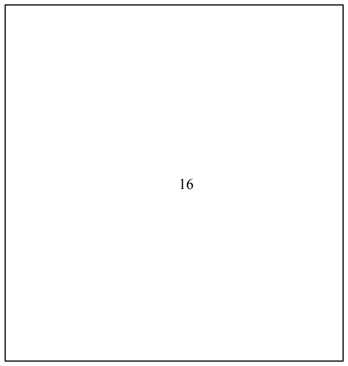
Eppstein's 16 element cube.
Isometric view



Top view.



Bottom view.



Conclusions

The construction given in this paper based on either the Four Split Pyramid or S. Mitchell's Geode-Template is easily programmable; the quality of the elements can be slightly improved by adding more elements; a gargantuan number of elements would be generated! There is a center point where with a weight of elements proportional to the number of quadrilaterals at the boundary. But, it can be remedied at the expense of adding more elements; the details of this operation are not discussed in this paper.

The construction to the solution Scott Mitchell's existence theorem presented to the constrained hexahedral problem is different from Scott Mitchell's proof besides the obvious; indeed, his solution cannot lead to the construction of this paper. In Scott Mitchell's general approach, a mesh that contains two loops with an odd of intersections will have the loops connected through an interior surface. The transition created from the ***Paired Partition*** eliminates all the self-intersecting loops in a non-differentiable manner! The construction is not one that untangles loops in the strict differentiable sense given by S. Smale's result.

From the construction, it follows that it is possible to generate a constrained hexahedral mesh with linear edges when dealing with a convex quadrilateral polyhedron. The question of finding a general construction with a minimal number of elements with linear edges is open.

Appendix

Connecting Cells from Figure 10

In this section, the rules on how to connect the elements of Figure 10 are discussed in more detail.

Given a quadrilateral mesh Q on the sphere and a minimal path of quadrilaterals connecting two given distinct quadrilaterals q and q' in Q The path will be composed by a sequence of quadrilaterals $\{q, q_1, q_2 ... q_{n-3}, q_{n-2}, q'\}$. The minimum path will have the property that no two vertices are traversed more than once, because otherwise, the portion between a vertex and its repetition in the sequence of vertices could be cut off to produce a smaller path. In addition, any two vertices in the path that are connected in the graph have to be successors and predecessors of each other in the path; otherwise, the path could be short-circuited in contradiction to the minimal property of the path. Hence the only two elements in the path that share elements with a given quadrilateral are the predecessor and the successor in the path, except for the first and last endpoints that do not have a predecessor or successor respectively.

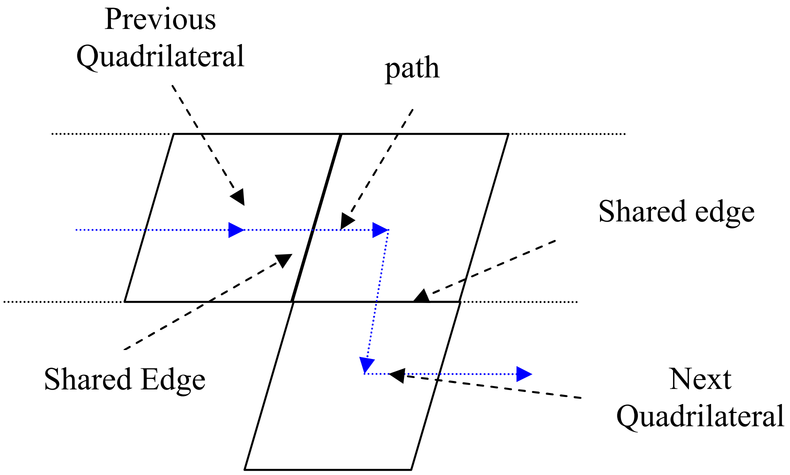
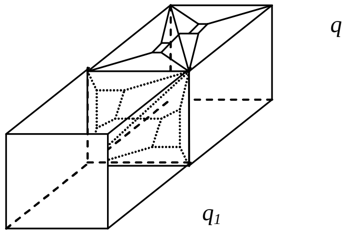


Figure 12

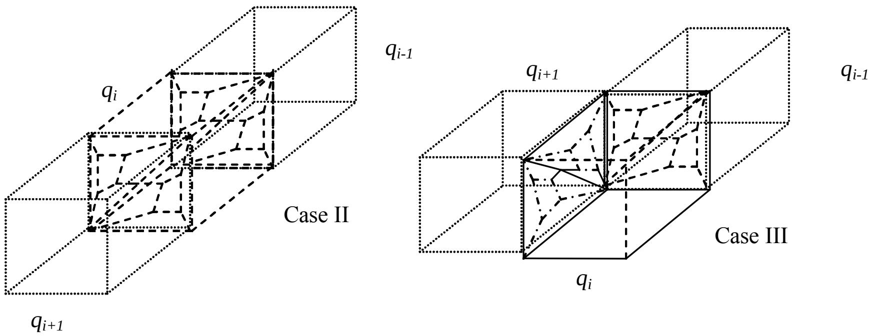
The first step in the construction is to place a cell of the type Case I at the first end point of the path q where one of the faces with ten quadrilaterals is pointing towards the center and the other towards q_1.

The ten quadrilateral face lies between q and q_1



A cell on top of element q_i will be either of type Case II or Case III depending on whether a the successor q_{i+1} is sitting opposite or adjacent to the edge between q and q_{i-1}.

Two possible cells for q_i depending on the location of q_{i+1} relative to q_{i-1}



The same reasoning applies when *i* is *n-2* depending on where *q'* lies; at *q'* cell of Case I is placed.

After placing the transition cells on the path, only the internal faces of the path have ten element quadrilateral cells except for the endpoints that have ten quadrilaterals pointing towards the center of the sphere; the exterior faces of the path are single quadrilaterals. On the quadrilaterals that lie outside the path, simple hexahedra or, equivalently, cells of case IV are placed to complete the transition to the new quadrilateral mesh. Every element in the initial quadrilateral mesh transitions to an element that is topologically the same with the exception of the elements *q* and *q'* that have transitioned to ten quadrilaterals each.

Geometric Placement of the Vertices

All the cells that have been used to introduce layers that lead to the transition to a ***Four Split*** mesh are convex cells. These cells are sitting on the cells composed by six quadrilaterals that constitute the resolution of self-intersecting loops illustrated in Figure 2. The interpolation of the geometric vertices will be done by connecting the points of the vertices at the corners of each cell to the center of the sphere. The geometric location of the vertices of each cell will be discussed below. The first layer of hexahedral elements that correspond to Figure 2 will be discussed in detail and the others with less detail because the basics do not differ from the first.

Point Interpolation for Paired Cells

Let *c* be the center of the sphere. Vertices v_7, v_8, v_9, and v_{10} are interpolated towards the center of the sphere by the following formulae:

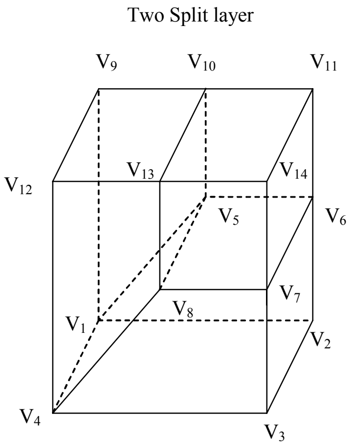
$$v_7 = \tfrac{1}{8}\, c + \tfrac{5}{8}\left(\tfrac{2}{3}\, v_2 + \tfrac{1}{3}\, v_3\right)$$
$$v_{10} = \tfrac{1}{8}\, c + \tfrac{5}{8}\left(\tfrac{2}{3}\, v_3 + \tfrac{1}{3}\, v_2\right)$$
$$v_8 = \tfrac{1}{2}\, v_7 + \tfrac{1}{2}\, v_2$$
$$v_9 = \tfrac{1}{2}\, v_{10} + \tfrac{1}{2}\, v_3$$

Point Interpolation for Two Split Cells

There will be two layers of ***Two Split*** cells: the first layer will be sitting on top of the quadrilaterals that result from the ***Paired Partition*** transition, and the second on top of the first one. In either case, the interpolation will be similar.

For a given quadrilateral with vertices $\{v_1, v_2, v_3, v_4\}$, add the layer of three hexahedral elements $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $\{v_1, v_5, v_8, v_4, v_9, v_{10}, v_{13}, v_{12}\}$, and $\{v_5, v_6, v_7, v_8, v_{10}, v_{11}, v_{14}, v_{13}\}$. The point's v_9, v_{11}, v_{14}, and v_{12} are interpolated by the formulae

$$v_5 = \tfrac{1}{4}(\tfrac{1}{4}\, c + 1\tfrac{5}{8}\, v_1 + 1\tfrac{5}{8}\, v_2)$$
$$v_6 = \tfrac{1}{2}\,(\tfrac{1}{8}\, c + 1\tfrac{5}{8}\, v_2)$$
$$v_7 = \tfrac{1}{2}\,(\tfrac{1}{8}\, c + 1\tfrac{5}{8}\, v_3)$$
$$v_8 = \tfrac{1}{4}(\tfrac{1}{4}\, c + 1\tfrac{5}{8}\, v_9 + 1\tfrac{5}{8}\, v_9)$$
$$v_9 = \tfrac{1}{8}\, c + \tfrac{5}{8}\, v_1$$
$$v_{10} = \tfrac{1}{2}\,(\tfrac{1}{4}\, c + \tfrac{5}{8}\, v_1 + \tfrac{5}{8}\, v_2)$$
$$v_{11} = \tfrac{1}{8}\, c + \tfrac{5}{8}\, v_2$$
$$v_{12} = \tfrac{1}{8}\, c + \tfrac{5}{8}\, v_4$$
$$v_{13} = \tfrac{1}{2}\,(\tfrac{1}{4}\, c + \tfrac{5}{8}\, v_3 + \tfrac{5}{8}\, v_4)$$
$$v_{14} = \tfrac{1}{8}\, c + \tfrac{5}{8}\, v_3$$

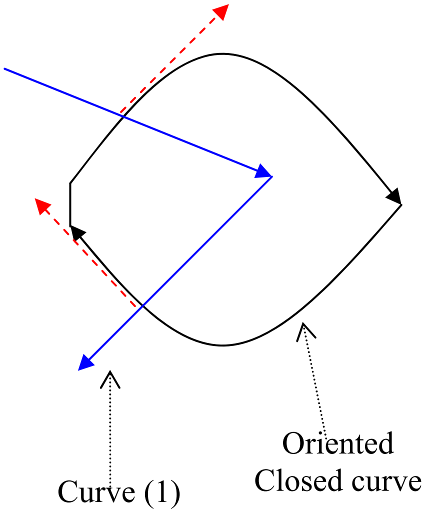


Simple Jordan Curves

The sphere minus a pole has a homeomorphism to the plane. A closed curve *C* in the sphere can be oriented if its homeomorphism can be oriented in the plane; a necessary and sufficient condition is that the curve has no self-intersections. The oriented curve divides the sphere into two regions: the interior lies to the left of the oriented curve, and the exterior to the right. This is the well known theorem on simple Jordan Curves [O'R1996: p2].

A collection of simple Jordan curves that do not intersect each other split the plane into regions that can be painted in black and white for example. Two regions with the same shade will not intersect, or, if the intersection is not empty, they will be the same regions. This fact allows the curves to be oriented such that the left sides of all the curves are pointing to the white region for example.

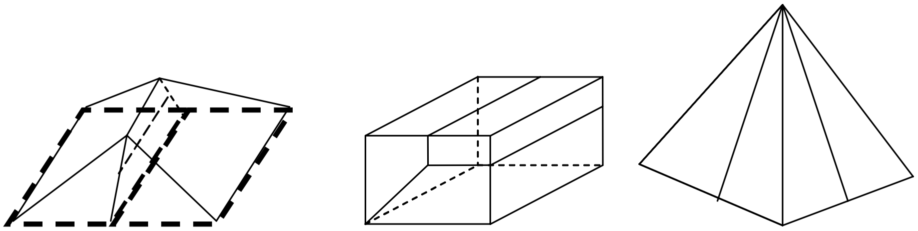
Suppose any curve intersects the closed curve and has no tangential intersections. If the points are sorted according the parameterization of the new curve, successive points will see the direction of the oriented curve going in opposite direction from each other. (Figure 13 illustrates this statement.) This fact was used to prove the existence of a transition to a quadrilateral mesh of four splits.



Curve (1) intersects the closed curve at two points. The tangents of at the two points are opposite to each other relative to the direction of curve (1). At one point the direction of the tangent at the closed curve is pointing to the left of the curve (1), and at the other point to the right.

Figure 13

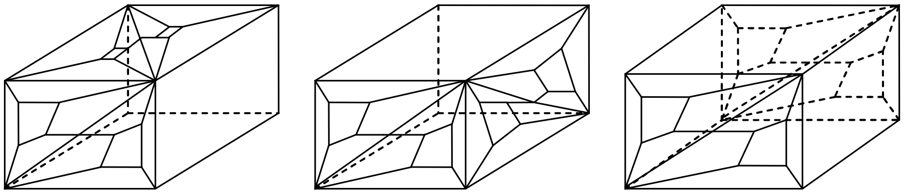
Supplementary Figures



1) Pairing layer

2) Two split layer

4) Four split pyramid layer contains four 180 degree quadrilaterals



3) Three types of cells needed to pair isolated quadrilaterals

Canonical Boundary Layers used through the construction

Figure 14

Various Interactions between Adjacent Paired Cells

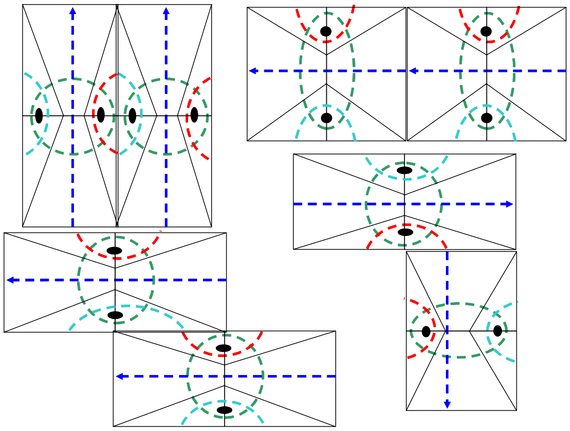


Figure 15

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