Approximate scale determination of the hydrogen atom from spin-orbit interaction

David C. Lush
davelush@comcast.net

(Dated: July 14, 2007)

We study the interaction of two point particles in a classical electrodynamics-like setting consisting of Maxwell's electrodynamics with an *ad hoc* incorporation of intrinsic spin and intrinsic magnetic moment. The effect of spin and intrinsic magnetic moments is modeled as a small deviation from nonrelativistic circular particle orbits due to Coulomb attraction. It is hypothesized that any possible non-radiative states of motion will have a necessary condition of mechanical angular momentum constancy, and an assessment is made of what orbital radii might achieve this. The interaction between spin and orbital angular momenta is examined as small precessional motions of the spins and orbits. It is shown, for perpendicular spin-orbit orientations, that there is only one orbital radius where a necessary condition for angular momentum constancy is approximately achievable. This orbital radius is shown to be one-sixteenth of the ground state atomic radius given by the Bohr model of hydrogen and as frequently used as a standard for atomic dimension. Possible implications of this result are discussed.

I. INTRODUCTION

In this work we revisit the question of whether the hydrogen atom can be usefully described as two point-like particles of suitable mass and electrical charge interacting via classical electrodynamics according to Maxwell. This model, where the particles traverse planetary orbit-like trajectories, is often called the Rutherford model.

In the modern view it is usually considered such an approach is doomed to failure due equally to the problems of radiative decay and viloation of energy coservation. The force of radiation reaction causes the electron to spiral in to the proton. One might argue however that these were not historically the primary reasons for abandoning this approach. Rather, there were two unrelated reasons less philosophical. First, there was an utter lack of success in discovering any possible explanation from classical electrodynamics for discrete allowed states of motion such as would be required to explain the discreteness of atomic spectra. Then, more importantly, a new theory came along that had great success in describing atomic phenomena of many sorts, including atomic spectra, in accurate detail. Although the new theory had the drawback of being postulated rather than derived, this seemed not significant. The theory worked. In contrast the earlier approach was not only understood to have philosophical problems with conservation of energy and radiative decay, but also had essentially no success whatsoever in explaining the existence of discrete spectra.

In the present day we know that there are important properties of subatomic particles other than mass and electric charge, that were not known when the classical-electrodynamics atomic models were initially attempted. In particular, subatomic particles of all sorts are now considered to have an intrinsic angular momentum, or spin, and associated with spin an intrinsic magnetic moment. Also, suggestively, the spin of subatomic particles involves directly Planck's constant which is also involved in discrete atomic spectra. One might wonder today what Bohr would have done with this knowledge in 1912.

Yet today we have still more knowledge to bring to bear on the problem that was not known in 1912. The prior work of Lorentz and Abraham on modeling the electron as a uniformly charged sphere led to a third-order equation of motion and runaway solutions. For most of the 20th Century this has been viewed as yet another justification for the abandonment of classical electrodynamics in the subatomic realm. Relatively recently however it has been proposed that these runaway states of motion may actually hold the basis for quantum mechanical behavior [1]. In the zitterbewegung interpretation of quantum mechanics, the spin is not an intrinsic property but rather a kinematical property that is a manifestation of the runaway motion. This gyrational motion provides plausible explanations for not only the meaning of the phase of the Shrodinger wave function, but also for how point-like particles might orbit without radiating [3]. However the mathematics involved in working out the dynamical details of such motions is prohibitively difficult at present, although some recent progress has been made [4],[5]. Determination of the exact motion requires the use of delay-differential equations, which do not admit readily of closed form solutions. Unique solutions do not follow from initial conditions specified at discrete instants of time. Rather, initial conditions must be supplied for the duration of the applicable delay. Effectively, one is left in a circular conundrum where in order to find the solution, one must know it already.

We hope from the above discussion we can draw sufficient justification to allow the reader to reconsider at least briefly what is according to present-day sensibilities a very naive approach to modeling the atom. We may have waiting in the wings a deeper new theory that is beyond our present ability to directly solve. Yet if we knew the solution we would be able to test it. What we need are ideas about what prospective solutions might look like. We believe the present approach can provide some.

II. SCALE INVARIANCE AND ANGULAR MOMENTUM CONSTANCY

One of the difficulties of constructing a useful atomic model from classical electrodynamics and point-like charged particles, lies in successfully determining the scale of atomic systems from a Coulomb interaction that is scale invariant, except for the effect of propagation delay [6]. Although accounting for propagation delay breaks the scale invariance, it would appear problematic to recover the scale of atomic phenomena, involving a particular distance scale as well angular momentum in small multiples of the reduced Planck's constant, from a delay depending only on distance and the velocity of light. These considerations however neglect a possible alternative scale-invariance-breaking phenomonology, due to the intrinsic magnetic moments of the particles. If intrinsic magnetic moments are considered in addition to Coulomb attraction, it is apparent that scale invariance is broken by the differing dependencies on interparticle separation of the electric and magnetic interactions. Also, the intrinsic magnetic moments directly involve Planck's constant. It might be considered natural to wonder therefore what are the implications of the known scale of atomic phenomena, in terms of the dynamics of the motion including effects of the presence of intrinsic magnetic moments and magnetic effects due to motion of charges as well as Coulomb attraction.

In the quantum mechanical treatment of, in the simplest case, the hydrogen atom, angular momentum is a constant of the motion for stable states. In the Shrödinger picture the stationary solutions of the wave equation are eigenvectors of the total angular momentum operator. The implications of this are quite rich, in that it results in a specific and somewhat elaborate set of rules for the allowed angular momenta of the atomic constituents, which is of course also thoroughly confirmed experimentally.

Apart from the quantum mechanical notion that stable states have constant angular momentum, it would seem reasonable to expect that any stable state of motion would have the property of angular momentum constancy as a necessary condition for being non-radiative.

Based on the above considerations, our starting point will be a hypothesis that angular momentum is a constant of the motion of stable atomic states. On the one hand, this will provide a useful basis for analysis and provide a powerful constraint on the possible motions. On the other hand, one may hope, it may point toward an alternative or deeper understanding of the basis of quantum theory.

In our analysis we shall assume that the hydrogen atom sole constituents are an electron and a proton, and that they each possess the accepted rest mass and electric charge values. Of equal importance, they each possess an intrinsic angular momentum, or spin, and an associated magnetic moment, with commonly accepted magnitudes.

III. INCORPORATION OF SPIN AND INTRINSIC MAGNETIC MOMENT

In Maxwell's electrodynamics the only sources of electromagnetic fields are electrical charges. Magnetic fields arise only due to motion of the charges. Due to the lack of a working model of the electron in Maxwell's theory, at least where the electron dimension is consistent with scattering experiments, our introduction of spin and an associated magnetic moment as intrinsic properties is entirely *ad hoc*.

In order to have magnetic field and spin as intrinsic properties, the electron and the distribution of its charge must have a nonzero size. Early attempts to model the electron as an extended object had problems enough prior to the deduction of the spin, however, and these were only further compounded by the necessity to represent the spin. It was realized that an actual rotating electron of accepted size would have to rotate faster than the speed of light. Nonetheless the idea of spin as an intrinsic property has persisted.

The existence of electron intrinsic angular momentum was deduced by Goudsmit and Uhlenbeck from spectrographic measurements, not by direct measurement. In stable states the electron must have a particular intrinsic magnetic moment magnitude in order to explain the Zeeman effect. This does not say much if anything about the constancy of the spin magnitude in other than stable states. We will nonetheless represent the spin as an intrinsic property similar to the electric charge where the magnitude is unvarying. We also assume the spin has a well-defined axis of rotation that can take on any orientation, and only one orientation at a time. However, in our work we do not intend to take a position about whether spin is fundamentally an intrinsic or a kinematical property of matter. Rather, we model it as an intrinsic property solely for expediency's sake. If the model provides new insights, the exercise will have been worthwhile.

The intrinsic angular momentum will thus be represented simply as

$$\boldsymbol{s} = \frac{\hbar}{2}\hat{\boldsymbol{s}} \tag{1}$$

where $\hbar$ is the reduced Planck's constant and $\hat{\boldsymbol{s}}$ is a unit-magnitude orientation vector. The intrinsic magnetic moments will be represented in the form

$$\boldsymbol{\mu} = \frac{ge}{2mc}\boldsymbol{s} = \frac{ge}{2mc}\frac{\hbar}{2}\hat{\boldsymbol{s}} \tag{2}$$

where the "g factor" has a fixed value depending on the particle type. For the electron

$$g = g_e \approx 2.0024 \approx 2 \tag{3}$$

and for the proton

$$g = g_p \approx 5.5856 \tag{4}$$

IV. IMPLICATIONS OF ANGULAR MOMENTUM CONSTANCY

In our model of the hydrogen atom we represent the electron and proton as dimensionless particles possessing mass, charge and intrinsic angular momentum and magnetic moments, orbiting around their common center of mass (COM). For convenience we assume the orbits are essentially circular, apart from small perturbations. (The details of the dynamics of the motions so far as we need are worked out in the Appendix.) The angular momentum of the bound-state hydrogen atom then consists of four components. These are the two orbital components due to the separate electron and proton orbital motions around the COM, and the two intrinsic angular momenta.

The orbital components are of the form

$$\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} \tag{5}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the electron or proton position and velocity in COM coordinates, and m is the applicable particle's mass.

When the orbiting particle has a charge then there is an associated orbital magnetic moment given by

$$\boldsymbol{m} = \frac{e}{2mc}\boldsymbol{L} \tag{6}$$

where e is the signed particle electrical charge, c is the speed of light.

In our atomic model, there are magnetic fields present due to intrinsic magnetic moments and due to the relative motion of the charged particles. The torque $\boldsymbol{N}$ on a magnetic dipole moment $\boldsymbol{m}$ in a magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B} \tag{7}$$

A torque applied to a magnetic moment will cause a change in the orientation of the moment according to the angular momentum associated with the magnetic moment. That is,

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{N} \tag{8}$$

where $\boldsymbol{l}$ may be either orbital or intrinsic angular momentum. For the case of orbital angular momentum, and using (6) and (7), (8) becomes

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{L} \times \frac{e\boldsymbol{B}}{2mc} \tag{9}$$

To begin assessing the implications of angular momentum constancy in our model, let us first consider a simplified case where only the electron has spin and where the proton orbital angular momentum may be neglected, as would be true for orbital motion in the limit of a very large proton mass compared to the electron mass. Angular momentum constancy will then require

$$0 = \frac{d\boldsymbol{J}}{dt} = \frac{d(\boldsymbol{L} + \boldsymbol{s})}{dt} \tag{10}$$

where $\boldsymbol{J}$ is the total angular momentum and $\boldsymbol{L}$ and $\boldsymbol{s}$ are the (electron) orbital and spin angular momenta.

Now, an equation of motion such as (9) is [15] (Goldstein, p 177) "exactly the equation of motion of a constant magnitude vector which is rotating in space about the direction $\boldsymbol{B}$ with an angular velocity" (Goldstein, p 177, eq 5-78)

$$\boldsymbol{\omega} = -\frac{e\boldsymbol{B}}{2mc} \tag{11}$$

It is therefore possible to rewrite (10), for constant spin and orbital angular momenta magnitudes, as

$$\boldsymbol{L} \times \boldsymbol{\omega}_L + \boldsymbol{s} \times \boldsymbol{\omega}_s = 0 \tag{12}$$

Now, if

$$\boldsymbol{\omega}_L = \pm\boldsymbol{\omega}_s \tag{13}$$

then

$$(\boldsymbol{L} \pm \boldsymbol{s}) \times \boldsymbol{\omega}_L = 0 \tag{14}$$

For the upper sign we have

$$\boldsymbol{J} \times \boldsymbol{\omega}_L = 0 \tag{15}$$

Or, $\boldsymbol{L}$ and $\boldsymbol{s}$ precess around $\boldsymbol{J}$ with equal angular velocity. For the lower sign, $\boldsymbol{L}$ and $\boldsymbol{s}$ precess around their vector difference. In either case, we must have

$$\omega_L = \omega_s \tag{16}$$

In an obvious way, including the proton orbital and spin angular momenta as well as the electron's angular momenta will lead to a similar result that constant angular momentum can be achieved, if all spin and orbital angular momentum magnitudes are constant, only if all individual precessional frequencies are of equal magnitude.

V. INTERACTION OF PROTON SPIN WITH ORBITING SPINLESS ELECTRON

As an initial illustration of how angular momentum constancy might lead to a preferred orbital radius we consider a simplified bound system where only the proton has spin and the proton orbital angular momentum may be considered negligible.

In a bound system consisting of an electron and a proton, where only the proton has intrinsic angular momentum and magnetic moment of the usual values, requiring that the total mechanical (i.e., non-electromagnetic) angular momentum be a constant of the motion will require that the precessional frequencies of the proton intrinsic and electron orbital angular momenta be identical. However, such will not generally be the case, even for circular orbits, for arbitrary orbital radius. It is natural to ask, therefore, at what radius or radii will a circular orbit possess the property that the mechanical angular momentum is a constant of the motion. For simplicity, we shall make an initial supposition that $\boldsymbol{L}$ and $\boldsymbol{s}$ are aligned approximately parallel, and consider only small precessional angles. In this case, the precessional frequencies will be very easy to calculate.

For the electron orbiting in the plane perpendicular to the direction of the proton spin, the magnetic field due to the proton intrinsic magnetic moment has a magnitude given by

$$B = \frac{g_p e\hbar}{4m_p c R^3} \tag{17}$$

The precessional angular velocity of the orbital angular momentum vector is then

$$\omega_L = \frac{e}{2m_e c}\frac{g_p e\hbar}{4m_p c R^3} = \frac{g_p e^2\hbar}{8m_p m_e c^2 R^3} \tag{18}$$

The magnetic field at the proton due to the electron orbital motion around the proton is

$$B = \frac{ev}{cR^2} \tag{19}$$

where the electron velocity is

$$v = \frac{e}{\sqrt{m_e R}} \tag{20}$$

The proton intrinsic angular momentum precessional frequency magnitude is then

$$\omega_s = \frac{eg_p}{2m_p c}\frac{e}{cR^2}\frac{e}{\sqrt{m_e R}} \tag{21}$$

or

$$\omega_s = \frac{g_p e^3}{2m_p c^2\sqrt{m_e}}R^{-5/2} \tag{22}$$

Equating precessional frequencies as required for angular momentum constancy yields

$$\frac{e}{2m_e c^2}\frac{g_p e\hbar}{4m_p R^3} = \frac{g_p e^3}{2m_p c^2\sqrt{m_e}}R^{-5/2} \tag{23}$$

or

$$R^{1/2} = \frac{e}{2m_e c^2}\frac{g_p e\hbar}{4m_p}\frac{2m_p c^2\sqrt{m_e}}{g_p e^3} \tag{24}$$

or

$$R^{1/2} = \frac{\hbar}{4m_e}\frac{\sqrt{m_e}}{e} \tag{25}$$

or

$$R = \left(\frac{\hbar}{4m_e}\right)^2\frac{m_e}{e^2} \tag{26}$$

or

$$R = \frac{1}{16}\frac{\hbar^2}{m_e e^2} \tag{27}$$

For comparison, the hydrogen ground-state radius in the Bohr model is given by

$$R_B = \frac{\hbar^2}{m_e e^2} \tag{28}$$

and so

$$R = \frac{1}{16}R_B \tag{29}$$

So, equating spin and orbital magnetic moment precessional frequencies for parallel, constant magnitude spin and orbital angular momenta has yielded a single electron-proton separation value where constancy of angular momentum is not precluded by differing precession frequencies. The formula for this separation value is similar to the equation for allowed range separation in the Bohr model of atomic hydrogen, differing by only a factor of one-sixteenth. However, (29) does not include the entire magnetic field experienced by the proton due to the electron as it does not include the magnetic field due to the electron intrinsic magnetic moment. The electron intrinsic magnetic field strength is not negligible compared to that due to the orbital motion at typical atomic separations, so it should be considered of interest to include it. We shall include this contribution in a subsequent section below. In any case, the example we have chosen in not the only case of interest, since as is well known the more important spin-orbit interaction in hydrogen is that between the electron spin and orbital magnetic moments. Quantum-mechanical analyses of this case typically are performed in a coordinate frame where the elctron is at rest, and where the electron spin interacts with the proton orbital motion in this electron rest frame. We shall therefore next attempt to assess the motions yielding angular momentum constancy under the interaction of the electron spin and proton orbit in the electron rest frame.

VI. ELECTRON REST FRAME INTERACTION OF ELECTRON SPIN WITH SPINLESS PROTON

In a bound system consisting of an electron and a proton, where only the electron has intrinsic angular momentum and magnetic moment of the usual values, requiring that the total mechanical angular momentum be a constant of the motion will require again that the precessional frequencies of the intrinsic and orbital angular momenta be identical. However, in this case, the situation is complicated by the fact that the electron orbital magnetic moment does not directly precess in an intrinsic magnetic moment. Rather, it is the proton orbital magnetic moment as defined in an electron-centered coordinate system, which precesses in the electron-centered coordinate system, and which, via a mechanism we shall describe in detail in a future work, manifests as an electron orbital moment precession in center-of-mass coordinates.

While it is true that the proton in reality does possess an intrinsic magnetic moment, its magnitude is much smaller than the electron magnetic moment and so may be neglected to a first order. As is well understood, the energetically more important spin-orbit interaction is that between the electron moments. The interaction between the electron's own moments as described above however must be considered somewhat indirect and so the treatment is more complicated than for the previous model.

We now consider the motion in a frame centered on the orbiting electron and moving with it. The particular frame under consideration is the result of a constant velocity translation relative to the proton-centered frame (or more properly the center-of-mass frame, but we ignore the distinction here), followed by an acceleration towards the proton in accordance with the acceleration of the electron in its orbit. In such a frame, the proton orbits the electron with a vector velocity opposite to the electron velocity in the COM frame.

The coordinate frame we consider here is identical to that described by Thomas in conjunction with his analysis of the spin-orbit interaction. It was shown by Thomas that the result of the successive Lorentz transformations from the COM to the electron-centered frame is equivalent to a rotation with frequency (Jackson, p 546, 11.119)

$$\boldsymbol{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{\boldsymbol{a}\times\boldsymbol{v}}{c^2} \tag{30}$$

We can thus relate the orbital angular momentum in the two frames [15] (Goldstein, p133, 4-102)

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_{COM} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{elec} + \boldsymbol{\omega}_T\times\boldsymbol{L} \tag{31}$$

or

$$\boldsymbol{L}\times\boldsymbol{\omega}_L = \boldsymbol{L}_p\times\boldsymbol{\omega}_{L_p} + \boldsymbol{\omega}_T\times\boldsymbol{L} \tag{32}$$

where $\boldsymbol{L}_p$ is the orbital angular momentum of the proton in the electron rest frame. In the electron rest frame, the proton velocity is the negative of the electron velocity in the laboratory frame, plus an additional component due to the Thomas precession of the electron rest frame. That is,

$$(\boldsymbol{v}_p)_{elec} = (-\boldsymbol{v}_e \pm R\omega_T\hat{\boldsymbol{v}}_e)_{COM} = -\boldsymbol{v}_e\left(1\pm\frac{R\omega_T}{v}\right) \tag{33}$$

so

$$\boldsymbol{L}_p = \boldsymbol{r}_p\times m_p\boldsymbol{v}_p = -\boldsymbol{r}_e\times -m_p\boldsymbol{v}_e\left(1\pm\frac{R\omega_T}{v}\right) \tag{34}$$

or

$$\boldsymbol{L}_p = \frac{m_p}{m_e}\left(1\pm\frac{R\omega_T}{v}\right)\boldsymbol{L} \tag{35}$$

or

$$\boldsymbol{L}_p \approx \frac{m_p}{m_e}\left(1\pm\frac{R}{v}\frac{v^3}{2Rc^2}\right)\boldsymbol{L} \tag{36}$$

or

$$\boldsymbol{L}_p \approx \frac{m_p}{m_e}\left(1\pm\frac{v^2}{2c^2}\right)\boldsymbol{L} \approx \frac{m_p}{m_e}\boldsymbol{L} \tag{37}$$

Eq (32) becomes

$$\boldsymbol{L}\times\boldsymbol{\omega}_L = \frac{m_p}{m_e}\boldsymbol{L}\times\boldsymbol{\omega}_{L_p} + \boldsymbol{\omega}_T\times\boldsymbol{L} \tag{38}$$

or

$$\boldsymbol{\omega}_L = \frac{m_p}{m_e}\boldsymbol{\omega}_{L_p} - \boldsymbol{\omega}_T \tag{39}$$

Similarly, for the electron spin

$$\left(\frac{d\boldsymbol{s}}{dt}\right)_{COM} = \left(\frac{d\boldsymbol{s}}{dt}\right)_{elec} + \boldsymbol{\omega}_T \times \boldsymbol{s} \tag{40}$$

and

$$\boldsymbol{s} \times \boldsymbol{\omega}_s = \boldsymbol{s} \times \boldsymbol{\omega}_{s'} + \boldsymbol{\omega}_T \times \boldsymbol{s} \tag{41}$$

or

$$\boldsymbol{\omega}_s = \boldsymbol{\omega}_{s'} - \boldsymbol{\omega}_T \tag{42}$$

where the prime indicates that the precessional frequency is the value observed in the electron rest frame.

Angular momentum constancy in the COM frame will require

$$0 = \left(\frac{d\boldsymbol{J}}{dt}\right)_{COM} = \left(\frac{d(\boldsymbol{L}+\boldsymbol{s})}{dt}\right)_{COM} \tag{43}$$

or

$$\boldsymbol{L} \times \boldsymbol{\omega}_L + \boldsymbol{s} \times \boldsymbol{\omega}_s = 0 \tag{44}$$

Angular momentum constancy will again have the necessary condition

$$\boldsymbol{\omega}_L = \pm \boldsymbol{\omega}_s \tag{45}$$

or, in either case,

$$\omega_L = \omega_s \tag{46}$$

which in turn yields,

$$\left|\frac{m_p}{m_e}\boldsymbol{\omega}_{L_p} - \boldsymbol{\omega}_T\right| = |\boldsymbol{\omega}_{s'} - \boldsymbol{\omega}_T| \tag{47}$$

In the present work we shall not attempt to investigate (47) in general. There are two problems inherent, one fundamental and one practical. The fundamental problem is that the motion of the orbital angular momentum vector is not strictly as given by (), which assumes a constant magnetic field, except in the case of spin orientaion perpendicular to the plane of the orbit. We can avoid both this problem and the practical problem of algebraic complexity by restricting our consideration for the present to the case of spin orientation perpendicular to the orbital plane. Then we will have that either

$$\frac{m_p}{m_e}\omega_{L_p} = \omega_{s'} \tag{48}$$

or

$$\frac{m_p}{m_e}\omega_{L_p} = \omega_{s'} + 2\omega_T \tag{49}$$

We shall first consider the case of (48).

In the electron rest frame, for the proton orbiting in the plane perpendicular to the direction of the electron spin, the magnetic field due to the proton intrinsic magnetic moment has a magnitude given by

$$B = \frac{g_e e\hbar}{4m_e cR^3} \tag{50}$$

The precessional angular velocity of the orbital angular momentum vector is then

$$\omega_{L_P} = \frac{e}{2m_p c}\frac{g_e e\hbar}{4m_e cR^3} = \frac{g_e e^2\hbar}{8m_p m_e c^2 R^3} \tag{51}$$

In the electron rest frame, the magnetic field at the electron due to the proton orbital motion around the electron is

$$B = \frac{ev}{cR^2} \tag{52}$$

where the proton velocity in the electron rest frame is

$$v = \frac{e}{\sqrt{m_e R}} \tag{53}$$

The electron intrinsic angular momentum precessional frequency magnitude in the electron rest frame, assuming the electron spin magnitude is the same in the electron frame as in the laboratory frame, is then

$$\omega_{s'} = \frac{eg_e B}{2m_e c} = \frac{eg_e}{2m_e c}\frac{e}{cR^2}\frac{e}{\sqrt{m_e R}} \tag{54}$$

or

$$\omega_{s'} = \frac{g_e e^3}{2c^2}{m_e}^{-3/2}R^{-5/2} \tag{55}$$

Equation (48) becomes

$$\frac{m_p}{m_e}\frac{g_e e^2\hbar}{8m_p m_e c^2 R^3} = \frac{g_e e^3}{2c^2}{m_e}^{-3/2}R^{-5/2} \tag{56}$$

or

$$\frac{1}{m_e}\frac{\hbar}{8m_e R^3} = \frac{e}{2}{m_e}^{-3/2}R^{-5/2} \tag{57}$$

or

$$\frac{\hbar}{8{m_e}^2} = \frac{e}{2}{m_e}^{-3/2}R^{1/2} \tag{58}$$

or

$$R^{1/2} = \frac{\hbar}{8{m_e}^2}\frac{2}{e}{m_e}^{3/2} \tag{59}$$

or

$$R^{1/2} = \frac{\hbar}{4e}{m_e}^{-1/2} \tag{60}$$

or

$$R = \frac{1}{16}\frac{\hbar^2}{m_e e^2} \tag{61}$$

or

$$R = \frac{1}{16}R_B \tag{62}$$

Interestingly, in spite of a large difference in the strength of the electron and proton intrinsic magnetic moments, we have arrived at the same expression as our previous analysis of spin-orbit interaction where only one of the particles possesses intrinsic spin and magnetic moment. However we have not yet considered the situation where the proton orbital angular momentum in the electron rest frame and the electron spin are antiparallel. For the circular orbit, the Thomas frequency may be evaluated from (30) as

$$\omega_T = \frac{\gamma^2}{\gamma+1}\frac{|\boldsymbol{a}\times\boldsymbol{v}|}{c^2} \tag{63}$$

or

$$\omega_T = \frac{1}{c^2}\frac{\gamma^2}{\gamma+1}\frac{v^3}{R} \tag{64}$$

or

$$\omega_T = \frac{1}{c^2}\frac{\gamma^2}{\gamma+1}\left(\frac{e}{\sqrt{m_e R}}\right)^3 R^{-1} \tag{65}$$

For the present application it is reasonable to approximate the gamma factor as unity so

$$\omega_T \approx \frac{1}{2c^2}e^3{m_e}^{-3/2}R^{-5/2} \tag{66}$$

Equation (49) becomes

$$\frac{m_p}{m_e}\frac{g_e e^2\hbar}{8m_p m_e c^2 R^3} = \frac{g_e e^3}{2c^2}{m_e}^{-3/2}R^{-5/2} + \frac{1}{c^2}e^3{m_e}^{-3/2}R^{-5/2} \tag{67}$$

or

$$\frac{g_e\hbar}{8{m_e}^2R^3} = e{m_e}^{-3/2}R^{-5/2}\left(\frac{g_e}{2}+1\right) \tag{68}$$

or

$$\frac{g_e\hbar}{8} = e{m_e}^{1/2}R^{1/2}\left(\frac{g_e}{2}+1\right) \tag{69}$$

or

$$R^{1/2} = \frac{g_e\hbar}{8e}\left(\frac{g_e}{2}+1\right)^{-1}{m_e}^{-1/2} \tag{70}$$

or

$$R^{1/2} \approx \frac{\hbar}{4e}\left(\frac{g_e}{2}+1\right)^{-1}{m_e}^{-1/2} \tag{71}$$

or

$$R^{1/2} \approx \frac{\hbar}{8e}{m_e}^{-1/2} \tag{72}$$

or

$$R \approx \frac{1}{64}R_B \tag{73}$$

We have not yet assessed the effect of the full magnetic field in neglecting the proton intrinsic magnetic moment, and so must return to this case to complete the analysis of the effect of the proton on the electron spin. The proton intrinsic magnetic moment however is much smaller than that of the electron and so the additional effect of the proton intrinsic magnetic moment on the electron spin precession is expected to be small. A proportionately larger effect of intrinsic magnetic moment can be expected from the electron intrinsic moment acting on the proton spin. We shall therefore next incorporate both particle intrinsic moments in that of the effects of the electron orbital and intrinsic magnetic moments on the proton spin.

VII. PROTON SPIN INTERACTION WITH ELECTRON ORBIT INCLUDING EFFECT OF ELECTRON SPIN ON PROTON SPIN

Our initial spin-orbit interaction analysis case was to consider the interaction of a proton possessing intrinsic spin and magnetic moments with a spinless electron. This yielded that orbital and spin precessional frequencies would be of equal magnitude at 1/16th the ground-state Bohr radius. In this section we will consider the effect on the proton precessional frequency of including the effect the electron intrinsic spin and, importantly, the associated electron intrinsic magnetic moment. The total magnetic field at the proton is of course that due to both the electron orbital motion and intrinsic magnetic moment, and so the proton spin precessional frequency neglecting the latter contribution cannot be exactly correct.

The total magnetic field at the proton due to both the electron orbital motion and intrinsic magnetic moment, for a circular orbit, for electron spin oriented perpendicularly to the orbital plane, is

$$B = \frac{g_e e\hbar}{4m_e c R^3} \pm \frac{e}{cR^2}\frac{e}{\sqrt{m_e R}} \tag{74}$$

where the upper sign corresponds to electron orbital and intrinsic angular momenta aligned parallel, and the lower sign, antiparallel. This may be rewritten as

$$B = \frac{e}{c} R^{-5/2}\left(\frac{g_e\hbar}{4m_e\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) \tag{75}$$

or

$$B = \frac{e}{c} {m_e}^{-1/2} R^{-5/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) \tag{76}$$

The proton precessional frequency is then

$$\omega = \frac{e g_p}{2m_p c}\frac{e}{c} {m_e}^{-1/2} R^{-5/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) \tag{77}$$

or

$$\omega = \frac{e^2 g_p}{2m_p c^2} {m_e}^{-1/2} R^{-5/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) \tag{78}$$

The electron orbit precessional frequency in the proton intrinsic magnetic moment is again

$$\omega_L = \frac{e}{2m_e c}\frac{g_p e\hbar}{4m_p c R^3} = \frac{g_p e^2\hbar}{8m_p m_e c^2 R^3} \tag{79}$$

Equating precessional frequencies:

$$\frac{e^2 g_p}{2m_p c^2} {m_e}^{-1/2} R^{-5/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) = \frac{g_p e^2\hbar}{8m_p m_e c^2 R^3} \tag{80}$$

or

$${m_e}^{-1/2} R^{-5/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) = \frac{\hbar}{4m_e R^3} \tag{81}$$

or

$$R^{1/2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right) = {m_e}^{1/2}\frac{\hbar}{4m_e} \tag{82}$$

or

$$R^{1/2} = {m_e}^{-1/2}\frac{\hbar}{4}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right)^{-1} \tag{83}$$

or

$$R = \frac{\hbar^2}{16m_e}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right)^{-2} \tag{84}$$

or

$$R = \frac{\hbar^2}{m_e e^2}\frac{e^2}{16}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right)^{-2} \tag{85}$$

or

$$R = R_B\left(\frac{4}{e}\right)^{-2}\left(\frac{g_e\hbar}{4\sqrt{m_e R}} \pm e\right)^{-2} \tag{86}$$

or

$$R = R_B\left(\frac{g_e\hbar}{e\sqrt{m_e R}} \pm 4\right)^{-2} \tag{87}$$

or

$$R\left(\frac{g_e\hbar}{e\sqrt{m_e R}} \pm 4\right)^{2} = R_B \tag{88}$$

or

$$R\left(\frac{{g_e}^2\hbar^2}{e^2 m_e R} \pm \frac{8g_e\hbar}{e\sqrt{m_e R}} + 16\right) = R_B \tag{89}$$

or

$$\frac{{g_e}^2\hbar^2}{e^2 m_e} \pm \frac{8g_e\hbar}{e\sqrt{m_e}}\sqrt{R} + 16R = R_B \tag{90}$$

This is a quadratic equation for the square root of R with coefficients

$$a = 16 \tag{91}$$

$$b = \pm\frac{8g_e\hbar}{e\sqrt{m_e}} \tag{92}$$

$$c = \frac{{g_e}^2\hbar^2}{e^2 m_e} - R_B \tag{93}$$

and solution

$$\sqrt{R} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{94}$$

where all four sign combinations are allowed to be considered. We then have

$$32\sqrt{R} = \mp\frac{8g_e\hbar}{e\sqrt{m_e}} \pm \sqrt{\left(\frac{8g_e\hbar}{e\sqrt{m_e}}\right)^2 - 64\left(\frac{{g_e}^2\hbar^2}{e^2 m_e} - R_B\right)} \tag{95}$$

or

$$4\sqrt{R} = \mp\frac{g_e\hbar}{e\sqrt{m_e}} \pm \sqrt{R_B} \tag{96}$$

or

$$4\sqrt{R} = \mp g_e\sqrt{R_B} \pm \sqrt{R_B} \tag{97}$$

or

$$4\sqrt{R} = \sqrt{R_B}(\mp g_e \pm 1) \tag{98}$$

or

$$R = \frac{1}{16}R_B(g_e \pm 1)^2 \approx \frac{1}{16}R_B, \frac{9}{16}R_B \tag{99}$$

The two solutions provided by Equation (124) correspond to the electron spin and orbital moments being antiparallel or parallel. Since the antiparallel case yields the same radius as previously where the proton intrinsic spin precesses at the same frequncy as the electron orbital moment, and since the calculation of the magnetic field at the electron is the same as the previous case omitting the contribution of the electron intrinsic moment, it is apparent that the magnetic field strength due to the electron intrinsic moment is twice that due to the electron orbital moment at a radius of one-sixteenth R_B.

VIII. ELECTRON REST FRAME INTERACTION OF ELECTRON SPIN WITH PROTON INCLUDING PROTON SPIN

In this section we consider again the interaction of the electron spin with the proton. This case will differ from our initial treatment of the electron spin interaction with the proton orbit by including the additional magnetic field component felt by the electron, due to the proton intrinsic magnetic moment.

The total magnetic field at the electron due to both the electron rest frame proton orbital motion and proton intrinsic magnetic moment, for a circular orbit, for electron spin oriented perpendicularly to the orbital plane, is

$$B = \frac{g_p e\hbar}{4m_p cR^3} \pm \frac{e}{cR^2}\frac{e}{\sqrt{m_e R}} \tag{100}$$

where the upper sign corresponds to proton orbital and intrinsic angular momenta aligned parallel, and the lower sign, antiparallel. This may be rewritten as

$$B = \frac{e}{c}R^{-5/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) \tag{101}$$

The electron precessional frequency is then

$$\omega = \frac{eg_e}{2m_e c}\frac{e}{c}R^{-5/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) \tag{102}$$

or

$$\omega = \frac{e^2 g_e}{2m_e c^2}R^{-5/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) \tag{103}$$

The proton orbit precessional frequency in the electron intrinsic magnetic moment is again

$$\omega_{L_p} = \frac{e}{2m_p c}\frac{g_e e\hbar}{4m_e cR^3} = \frac{g_e e^2\hbar}{8m_e m_p c^2 R^3} \tag{104}$$

Equation () becomes:

$$\frac{e^2 g_e}{2m_e c^2}R^{-5/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) = \frac{m_p}{m_e}\frac{g_e e^2\hbar}{8m_e m_p c^2 R^3} \tag{105}$$

or

$$R^{-5/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) = \frac{\hbar}{4m_e R^3} \tag{106}$$

or

$$R^{1/2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right) = \frac{\hbar}{4m_e} \tag{107}$$

or

$$R^{1/2} = \frac{\hbar}{4m_e}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right)^{-1} \tag{108}$$

or

$$R = \frac{\hbar^2}{16{m_e}^2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right)^{-2} \tag{109}$$

or

$$R = \frac{\hbar^2}{m_e e^2}\frac{e^2}{16m_e}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right)^{-2} \tag{110}$$

or

$$R = R_B\left(\frac{4\sqrt{m_e}}{e}\right)^{-2}\left(\frac{g_p\hbar}{4m_p\sqrt{R}} \pm \frac{e}{\sqrt{m_e}}\right)^{-2} \tag{111}$$

or

$$R = R_B\left(\frac{g_p\sqrt{m_e}\hbar}{m_p e\sqrt{R}} \pm 4\right)^{-2} \tag{112}$$

or

$$R\left(\frac{g_p\sqrt{m_e}\hbar}{m_p e\sqrt{R}} \pm 4\right)^{2} = R_B \tag{113}$$

or

$$R\left(\frac{{g_p}^2 m_e\hbar^2}{{m_p}^2 e^2 R} \pm \frac{8g_p\sqrt{m_e}\hbar}{m_p e\sqrt{R}} + 16\right) = R_B \tag{114}$$

or

$$\frac{{g_p}^2 m_e\hbar^2}{{m_p}^2 e^2} \pm \frac{8g_p\sqrt{m_e}\hbar}{m_p e}\sqrt{R} + 16R = R_B \tag{115}$$

This is a quadratic equation for the square root of R with coefficients

$$a = 16 \tag{116}$$

$$b = \pm\frac{8g_p\sqrt{m_e}\hbar}{m_p e} \tag{117}$$

$$c = \frac{{g_p}^2 m_e\hbar^2}{{m_p}^2 e^2} - R_B \tag{118}$$

and solution

$$\sqrt{R} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{119}$$

where all four sign combinations are allowed to be considered. We then have

$$32\sqrt{R} = \mp\frac{8g_p\sqrt{m_e}\hbar}{m_p e} \pm \sqrt{\left(\frac{8g_p\sqrt{m_e}\hbar}{m_p e}\right)^2 - 64\left(\frac{{g_p}^2 m_e\hbar^2}{{m_p}^2 e^2} - R_B\right)} \tag{120}$$

or

$$4\sqrt{R} = \mp\frac{g_p\sqrt{m_e}\hbar}{m_p e} \pm \sqrt{R_B} \tag{121}$$

or

$$4\sqrt{R} = \mp\frac{g_p m_e}{m_p}\sqrt{R_B} \pm \sqrt{R_B} \tag{122}$$

or

$$4\sqrt{R} = \sqrt{R_B}(\mp\frac{g_p m_e}{m_p} \pm 1) \tag{123}$$

or

$$R = \frac{1}{16}R_B(\frac{g_p m_e}{m_p} \pm 1)^2 \approx \frac{1-\epsilon}{16}R_B, \frac{1+\epsilon}{16}R_B \tag{124}$$

with

$$\epsilon = \frac{2g_p m_e}{m_p} \approx 0.006 \tag{125}$$

IX. DISCUSSION

In Equations (29) and (62), we have shown that for two different types of spin-orbit interaction, for similar relative orientation between spin and orbital angular momenta, and making similar simplifying assumptions that only one of the particles has an intrinsic magnetic moment, the same unique electron-proton separation value results where a necessary condition for angular momentum constancy is met. This value is similar in appearance to the Bohr ground state radius, though smaller by a factor of 16. These initial analyses however neglect the spin-on-spin interaction, and including it in one case led to an allowed radius that was 9/16 the Bohr radius.

At this point it is justified to observe that at the least we have determined an atomic scale involving Planck's constant to the proper power, as well as the electron mass and charge to proper powers, and where Planck's constant has entered not in an *ad hoc* fashion but rather through its association with intrinsic spin. While our hypothesis that preferred motions are those with constant angular momentum might itself be considered *ad hoc*, we would argue that this is inherently more interesting and admitting of further investigation of its basis than the conventional quantum mechanical assertion that angular momentum may be exchanged only in integer multiples of h-bar. We propose, time and further investigation may determine that it is not *ad hoc* at all.

Finally, we observe that our approach is interestingly consistent with the Bohr Correspondence Principle, whereby quantum physics transitions to classical physics in the limit of large quantum numbers. The effects on the motion due to intrinsic magnetic moments decrease with separation faster than the inverse square law of the Coulombic interaction, and so it is easy to see (and prove) that they will be insignificant at the same ranges where it is known that classical electrodynamics and Coulomb attraction can accurately describe atomic phenomena.

X. AN ALTERNATIVE METHOD FOR MODELING SPIN-ORBIT INTERACTIONS

Our modeling of spin-orbit interaction as precession of magnetic dipole moments in constant magnetic fields was seen to be limited from the start. It is valid only for infinitesimal precessional motions around constant magnetic field vectors. An alternative approach is to determine the induced fields at each particle due to the intrinsic magnetic moment motion as well as due to its position [12]. The motion of the intrinsic moment in general results in an electric field and resulting electric force due to particle charge. The force on the proton due to the motion of electron intrinsic magnetic moment, for an electron orbiting circularly at about a Bohr radius, is much weaker than the Coulomb attraction between the particles but much stronger than the radiation reactive force on the electron due to its orbital motion. (It is shown in [12] that propation delay of the force due to intrinsic moment motion results in a nonradial component on the proton that is equal to the radiation reaction on the electron at 9/16 Bohr radius.) These forces even neglecting delay introduce significant new dynamics beyond the Keplerian-orbit model. It is possible to understand the electron spin interaction with the electron's own orbit based on this approach, because of the force due to intrinsic moment motion action on the proton and the resulting motion of the proton affecting the electron orbit through the Coulomb binding force. This interaction does not in general preserve the mechanical angular momentum nor the linear momentum of the center of mass. Attempts to determine what specific states of motion might exhibit angular momentum constancy are the object of ongoing efforts.

XI. POSSIBLE EXTENSIONS OF THE MODEL

Appel and Kiessling have developed a rigorous electrodynamics incorporating spin in [13]. Using their formulation as a basis for modeling will not only add rigor to the present approach, it will incorporate the possibility of the spin magnitude, as well as orientation, to participate in the dynamics, while preserving the simplicity of working in the Galilean limit.

XII. CONCLUSION

It has been shown that including the effects of particle intrinsic magnetic moments in a classical-electrodynamics atomic model, along with a hypothesis that the total angular momentum of stable states will be a constant of the motion, leads to particular allowed atomic radii that are similar in magnitude to the radius of the hydrogen ground state as given by the Bohr model.

XIII. ACKNOWLEDGEMENT

The author thanks Professor Jayme De Luca for reviewing and commenting on an earlier and substantially different version of this paper, and for subsequent discussions. The suggestion of the use of the work of Kiessling and Appel is his. Any errors of reasoning or mathematics are solely the fault of the author, however.

XIV. APPENDIX: OVERVIEW OF THE CLASSICAL AND BOHR HYDROGEN ATOM MODELS

The magnitude of the Coulomb force, F, acting between two charged particles of equal charge magnitude, e, and separated by a distance R, is, in Gaussian units,

$$F = \frac{e^2}{R^2} \tag{126}$$

For particle charges of opposite polarity the Coulomb force is attractive and, neglecting effects due to propagation delay, is oppositely directed for each particle, so we can write

$$\boldsymbol{F}_e = -\boldsymbol{F}_p \tag{127}$$

where $\boldsymbol{F}_e$ and $\boldsymbol{F}_p$ are the vector Coulomb forces on the electron and proton.

It is well known that a force law such as that given by (127) results in conservative motion, and for the case of inverse-square-law force such as given by (126), known as Kepler's problem, results in regular elliptical orbits. In this work we confine our attention to the case of circular orbits, however.

We suppose the electron and proton are in a circular orbit around their common center of mass. Then balancing the centrifugal acceleration of the electron with the Coulomb attraction from the proton yields

$$m_e \frac{{v_e}^2}{R_e} = \frac{e^2}{R^2} \tag{128}$$

where m_e and v_e are the electron mass and velocity measured in the center-of-mass frame, R_e is the electron distance from the center of mass and R is the electron-proton separation. We also have that

$$R = R_e + R_p \tag{129}$$

where R_p is the proton distance from the center of mass, and the definition of the center of mass

$$m_e R_e = m_p R_p \tag{130}$$

so

$$R = \left(1 + \frac{m_e}{m_p}\right) R_e = \left(\frac{m_p + m_e}{m_p}\right) R_e \tag{131}$$

or

$$R_e = \left(\frac{m_p}{m_p + m_e}\right) R \tag{132}$$

Also, in order for the electron and proton to remain opposite each other across the center of mass, we must have that

$$\frac{\boldsymbol{v}_e}{R_e} = -\frac{\boldsymbol{v}_p}{R_p} \tag{133}$$

so

$$\boldsymbol{v}_p = -\frac{m_e}{m_p} \boldsymbol{v}_e \tag{134}$$

If $\boldsymbol{v}$ is the electron velocity relative to the proton, then

$$\boldsymbol{v} = \boldsymbol{v}_e - \boldsymbol{v}_p \tag{135}$$

or

$$\boldsymbol{v} = \boldsymbol{v}_e + \frac{m_e}{m_p} \boldsymbol{v}_e = \left(\frac{m_p + m_e}{m_p}\right) \boldsymbol{v}_e \tag{136}$$

Substituting (132) into (128):

$$m_e \frac{m_p + m_e}{m_p} \frac{{v_e}^2}{R} = \frac{e^2}{R^2} \tag{137}$$

and with (136)

$$m_e \frac{m_p + m_e}{m_p} \left(\frac{m_p}{m_p + m_e}\right)^2 \frac{v^2}{R} = \frac{e^2}{R^2} \tag{138}$$

or

$$\frac{m_e m_p}{m_p + m_e} \frac{v^2}{R} = \frac{e^2}{R^2} \tag{139}$$

And so

$$v^2 = \frac{e^2}{m_r R} \tag{140}$$

where

$$m_r = \frac{m_e m_p}{m_p + m_e} \tag{141}$$

is the "reduced" mass of the electron.

The electron velocity as measured in an inertial reference frame with origin at the center of mass between the particles is thus

$$v_e = \frac{m_p}{m_p + m_e} \frac{e}{\sqrt{m_r R}} \tag{142}$$

or

$$v_e = \frac{m_p}{m_p + m_e} \sqrt{\frac{m_p + m_e}{m_p}} \frac{e}{\sqrt{m_r R_e}} \tag{143}$$

or

$$v_e = \sqrt{\frac{m_p}{m_p + m_e}} \sqrt{\frac{m_p + m_e}{m_e m_p}} \frac{e}{\sqrt{R_e}} \tag{144}$$

or

$$v_e = \frac{e}{\sqrt{m_e R_e}} \tag{145}$$

The proton velocity as measured in an inertial reference frame with origin at the center of mass between the particles is

$$v_p = \frac{m_e}{m_p} v_e = \frac{m_e}{m_p} \frac{e}{\sqrt{m_e R_e}} = \frac{\sqrt{m_e}}{m_p} \frac{e}{\sqrt{R_e}} \tag{146}$$

or

$$v_p = \frac{e\sqrt{m_e}}{m_p} {R_e}^{-1/2} = \frac{e\sqrt{m_e}}{m_p} \left(\frac{m_p}{m_e} R_p \right)^{-1/2} = \left(\frac{m_e}{m_p} \right) \frac{e}{\sqrt{m_p R_p}} \tag{147}$$

The ground-state radius from the Bohr atomic theory for hydrogen is given by

$$R_B = \frac{\hbar^2}{m_e e^2} \tag{148}$$

where $\hbar$ is the reduced Planck's constant.

[1] David Hestenes, *Spin and uncertainty in the interpretation of quantum mechanics*, Am. J. Phys. 47(5), May 1979

[2] David Hestenes, *The Zitterbewegung Interpretation of Quantum Mechanics*, Found. Physics. 20(10), (1990) 1213-1232

[3] David Hestenes, *Zitterbewegung in Radiative Processes*, D. Hestenes and A Weingartshofer (eds.), *The Electron*, 21-36, 1991

[4] Jayme De Luca, *Stiff three-frequency orbit of the hydrogen atom*, Phys Rev E 73, 026221, Feb 2006

[5] Jayme De Luca, *Geometric integration of the electromagnetic two-body problem*, J. Math. Phys. 48, 012702, Jan 2007

[6] Jayme De Luca, *Electrodynamics of a two-electron atom with retardation and self-interaction*, Phys Rev E 58(5), Nov 1998

[7] Uhlenbeck and Goudsmit, *TBD*, TBD, 1925

[8] Goudsmit, *TBD*, TBD, 1973

[9] Stern and Gehrlach, *TBD*, TBD, 1926

[10] F. Rohrlich, *Classical Charged Particles*, TBD, 2007

[11] L.H. Thomas, *TBD*, TBD, 1926

[12] David Lush, *TBD*, Unpublished, 2007, available at

[13] Walter Appel and Michael K.-H. Kiessling, *Mass and Spin Renormalization in Lorentz Electrodynamics*, arXive:math-ph/0009003v2 8 Oct 2000

[14] J. D. Jackson, *Classical Electrodynamics*

[15] H. Goldstein, *Classical Mechanics*