Preface

(from "Nonlinear Difference Equations" by Hassan Sedaghat)

It is generally acknowledged that deterministic formulations of dynamical phenomena in the social sciences need to be treated differently from similar formulations in the natural sciences. Social science phenomena typically defy precise measurements or data collection that are comparable in accuracy and detail to those in the natural sciences. Consequently, a deterministic model is rarely expected to yield a precise description of the actual phenomenon being modelled. Nevertheless, as may be inferred from a study of the models discussed in this book, the qualitative analysis of deterministic models has an important role to play in understanding the fundamental mechanisms behind social science phenomena. The reach of such analysis extends far beyond technical clarifications of classical theories that were generally expressed in imprecise literary prose.

The inherent lack of precise knowledge in the social sciences is a fundamental trait that must be distinguished from "uncertainty." For instance, in mathematically modelling the stock market, uncertainty is a prime and indispensable component of a model. Indeed, in the stock market, the rules are specifically designed to make prediction impossible or at least very difficult. On the other hand, understanding concepts such as the "business cycle" involves economic and social mechanisms that are very different from the rules of the stock market. Here, far from seeking unpredictability, the intention of the modeler is a *scientific* one, i.e., to clarify and explain the phenomenon in an objective way that will make it possible to apply what is learned (e.g., to moderate down-turns or busts, and thus help lessen human suffering).

Although undesirable in a scientific study of the business cycle and similar concepts, uncertainty is impossible to avoid completely in the social sciences. For this reason, there is considerable room for stochastic formulations in mathematical models, as long as one is careful to *not* attribute complex, but deterministic phenomena to random effects. For instance, it is known that the mathematical equations that result in persistent oscillatory behavior that is asymptotically and structurally stable *cannot* be linear (see Section 4.1 below). Thus, in part to assure that a system of *linear* equations generated persistent oscillations of the type seen in the economic data, it was common in the 1960's and '70's to add stochastic terms (called "random shocks") to the linear equations. But this was tantamount to *assuming* that the economic mechanisms behind the business cycle were incapable of generating sustained oscillations. Such an assumption never had any support, either philosophically or on economic grounds. Moreover, its tacit acceptance provided no credible information about the economic reality behind the business cycle - almost anything could be attributed to random effects when linear oscillations dissipated. A similar comment applies to other social science models, and the "knowledge" gained about them through statistical analyses based on linear equations. A more realistic, and informative, approach would be to propose a deterministic model whose properties can be analyzed qualitatively, and then *add* stochastic terms to the deterministic formalism for the sake of better fitting the existing data, if any.

The lack of precision in the social sciences that was noted above has important consequences for both the modelling, and the associated mathematical theory. In particular, if a model is specified by a mapping F of the Euclidean space $\mathbb{R}^m$, then typically F is to be given by means of the various properties that it is supposed to have, rather than by specific analytical expressions for which no satisfactory empirical justification can be found. The properties of F are generally deduced from the semantic context of the model through a process of abstraction that has matured considerably in recent times. Most (though not all) of the mathematical models encountered in this book involve *partially specified* mappings that reflect the characteristic coarseness of models. The mathematical results about such mappings typically involve rigorously establishing such qualitative properties as permanence and boundedness, persistent oscillations (periodic or aperiodic), sensitivity to initial conditions, and stability, instability, and the global attractivity of equilibria and cycles.

This monograph is split into two parts: The first gives a rigorous yet general mathematical treatment of maps and equations (Chapters 2-4) containing both some of the best known results in the literature, and many results that are quite recent (including a few hitherto unpublished ones). Several new concepts such as semiconjugates, polymodal systems and ejector cycles, persistent oscillations and absorbing intervals, as well as recently developed analytical techniques are presented in this book, many for the first time. The choice of topics in Chapters 2-4 is motivated to a large extent by the preceding observations about social science models and the formalism that seems appropriate to them. The mathematical treatment is rigorous, and the vast majority of theorems come with complete proofs. For readers who do not wish to study the sometimes long and technical proofs, a liberal supply of corollaries, remarks and examples provide a good sense of the boundaries of the main results, i.e., their applications and their limitations.

In Chapter 2, the theory on the real line is studied. Major topics include necessary *and* sufficient conditions for the asymptotic stability and instability of fixed points, coexistence and the Sharkovski ordering of cycles, the Singer-Allwright theory of limit cycles and the modern theory of one-parameter bifurcations. Chapter 3 begins with a presentation of LaSalle's approach to Liapunov stability for maps, and then proceeds to a discussion of mappings of Euclidean spaces that are semiconjugate to maps of the real line. Semiconjugacy as presented here, extends the notion of invariants to more general mappings and is intimately related to Liapunov functions. Further, it permits the extension of certain topics from the real line to the higher dimensional context. For example, it is possible to obtain extensions of the Li-Yorke Theorem that are very different from known analogs such as Marotto's Theorem. The last section of Chapter 3 is concerned with a light-hearted, though systematic study of complex threshold systems as special types of piecewise linear or polymodal systems. Certain constructs, such as ejector cycles, are seen to allow an application of the continuous theory to obtaining results about the global behavior of trajectories. Chapter

4 is concerned with the relatively better developed topic of higher order, scalar difference equations and here the level of rigor is proportionately higher.

The second half of the book (Chapters 5,6) presents several models from economics and other social sciences. In Chapter 5, a few of these models have been rigorously (though often not exhaustively) analyzed, following a brief description and derivation for each model. Readers who are interested in semantic aspects beyond what is presented here will find it more useful to learn them first hand from the original authors who introduced the models. Chapter 6 contains more models, each of which is presented in a brief format. In some cases, the mathematical analysis is similar to what is found in Chapter 5, and in these cases, it is left to the reader to complete the tasks. In the remaining cases, the analysis either requires material that goes beyond the scope of this monograph, or there is no known analytical work beyond what is provided by their authors' (usually in the form of numerical simulations). In a few of these latter models, the fundamental equations may need to be modified or restricted so as to permit the existence of an adequate number (e.g., an open set) of bounded, semantically viable trajectories within the positive cone of $\mathbb{R}^m$.

This book has been written in such a way that it can be read and the gist of it understood not only by professional mathematicians, but also by readers with limited expertise (or interest) in rigorous proofs. Such readers may wish to study the statements of theorems, and then proceed to study their consequences in various examples and models. The flavor of the book is mathematical however; it is assumed that the typical reader has been exposed to college undergraduate-level mathematics, and that he or she has gained some appreciation for precise mathematical analysis as a result. The rigor and universal validity of the mathematical language permits an objective discussion of scientific concepts and theories, and as such it is an invaluable tool for developing a scientific understanding of the complex and surprisingly non-random area covered by the social science models.