Section 2.1B: Asymptotic Stability.

The next theorem (and the main result of this section) states conditions equivalent to that in Theorem 2.1.1. The key idea in Theorem 2.1.1 was that *the left half and the right half of f must strike a balance for $\bar{x}$ to be asymptotically stable; if one half is rather steep, the other half must be flat enought that the combined action of the two halves will be subdued.* Since the combined action of the two halves amounts to the action of f^2, the appearence of f^2 in Part (b) of Theorem 2.1.2 below is expected. Condition (d) gives a more intutitive interepretation of Theorem 2.1.1, and (e) suggests how to get around cases where a direct inversion of f_r is difficult.

Theorem 2.1.2. (Asymptotic Stability) *Let $\bar{x}$ be a fixed point of f. The following statements are equivalent*:

(a) *$\bar{x}$ is asymptotically stable*;

(b) *There is a proper I-neigborhood U of $\bar{x}$ on which the following inequality holds*:

$$[f^2(x) - x](x - \bar{x}) < 0, \qquad x \neq \bar{x},\ x \in U \subset I \tag{T2.1.2a}$$

(c) *There is a proper I-neigborhood U of $\bar{x}$ on which* (**??**) *holds*;

(d) *There is a proper I-neighborhood U of $\bar{x}$ such that*:

$$[f(x) - x](x - \bar{x}) < 0, \qquad x \neq \bar{x},\ x \in U \subset I \tag{T2.1.2b}$$

and over $U_l - \{\bar{x}\}$, the graph of f_r^{-1} lies above the graph of f;

(e) *Inequality* (T2.1.2b) *holds on U, and for the parametrizations of f_l and f_r^{-1} given by $(x_l(t), y_l(t))$ and $(y_r(s), x_r(s))$ respectively, it is ture that*:

$$x_l(t) = y_r(s) \quad \text{implies} \quad x_r(s) > y_l(t).$$

(f) *There is a proper I-neighborhood U of $\bar{x}$ such that for $x \in U - \{\bar{x}\}$*,

$$(\phi(x) - f(x))(\phi(x) - x)(f(x) - x)\,(x - \bar{x}) < 0. \tag{T2.1.2c}$$

Proof. We show that (a)⇒(b)⇒(c)⇒(a) and (c)⇒(d)⇒(e)⇒(f)⇒(c). First, if (a) is true, then there is a sufficiently small I-neighborhood U of $\bar{x}$ on which the equality $f^2(x) = x$ holds only at $\bar{x}$ and every point in U is attracted to $\bar{x}$. Hence, the continuous function $f^2(x) - x$ either does not change its sign over U, or if it does, then the sign change can occur only at $\bar{x}$. If $f^2(x) > x$ for all $x \in U_r^o$, and $x_0 \in U_r^o$, then $f^2(x_0) > x_0$; if $f^2(x_0) \in U_r$ also, then another application of f^2 leads further away from $\bar{x}$ and the process continues until the trajectory $\{f^{2n}(x_0)\}$ exits U_r, no matter how close x_0 is to $\bar{x}$. Thus $\bar{x}$ cannot be stable, contradicting (a). Similarly, (a) is contradicted if $f^2(x) < x$ for all $x \in U_l^o$. Now (T2.1.2a) follows and (b) is established.

Next, suppose that (b) is true. Then (T2.1.2b) must hold, since otherwise there is either a fixed point other than $\bar{x}$ in U at which (b) would be false, or else, $f(x) < x$ (respectively, $f(x) > x$) for all $x < \bar{x}$ (respectively, $x > \bar{x}$) in U, in which case choosing x_0 sufficiently close to $\bar{x}$ so that $f(x_0) \in U$ implies that $f^2(x_0) < x_0$ (respectively, $f^2(x_0) > x_0$) also, again contradicting (b).

To establish (c), it remains to show that $\phi(x) > f_l(x)$ for $x \in U_l$. This is clear if $\phi(x) \geq a > 0$ for all $x \in U_l^o$; otherwise, arguing as in the last two cases in the proof of Theorem 2.1.1, we conclude that there is either a sequence of period-2 points converging to $\bar{x}$ from the left, or else, there is $x' \in U_l^o$ close to $\bar{x}$ such that $x_0 = f(x') \in U_r$ and $f^2(x_0) \in U_r$ with $f^2(x_0) > x_0$. Since in either case (b) is contradicted, we must assume that (c) holds. Finally, in Theorem 2.1.1 it was established that (c) implies (a).

Next, note that (d) follows easily from (c) because conditions (**??**) imply (T2.1.2b), and by Lemma 2.1.3(c) ϕ (hence also the graph of f_r^{-1}) dominates f on U_l^o if and only if ϕ dominates μ, hence also f_l. In light of Lemma 2.1.3(c), (e) is just a rephrasing of (d), hence equivalent to it. Statement (f) is an immediate consequence of (e), or equivalently (d), which implies that $\phi(x) > f(x)$ for all $x \in U - \{\bar{x}\}$ (for $x > \bar{x}$, the graph of f_r^{-1} lies above the identity line if and only if f_r lies below that line).

Finally, assume (f) holds. For $x < \bar{x}$, $\phi(x) > \bar{x} > x$, so if $\phi(u) < f(u)$ for some $u < \bar{x}$, then $f(u) > \bar{x} > u$ and (T2.1.2c) fails. Hence $\phi(x) > f(x)$ for all x, and so by (T2.1.2c) $f(x) > x$. For $x > \bar{x}$, the product $(\phi(x) - f(x))(\phi(x) - x)$ is always positive, since both f and the identity line always lie on the same side of f_r^{-1}. Therefore, by (T2.1.2c) $f(x) - x < 0$ and condition (**??**) is established.

Example 2.1.2. Consider the continuous mapping

$$f(x) = xe^{4(1-x)/(1+x)}, \quad x \geq 0.$$

The set of all non-negative real numbers $[0, \infty)$ is invariant under f, which has a unique positive fixed point $\bar{x} = 1$, with $f'(1) = -1$. We establish the asymptotic stability of $\bar{x}$ using Theorem 2.1.2(e) as follows. For $x \geq 0$, parametrize f by setting $t = -(1-x)/(1+x)$, and get the representations:

$$f_l: \; x_l(t) = \frac{1+t}{1-t}, \; y_l(t) = \frac{1+t}{1-t}e^{-4t}, \quad -1 \leq t < 0$$

and

$$f_r^{-1}: \; y_r(s) = \frac{1+s}{1-s}e^{-4s}, \; x_r(s) = \frac{1+s}{1-s}, \quad 0 < s < 1.$$

Straightforward calculation now shows that the conditions in Theorem 2.1.2(e) are satisfied for all $t \in [-1, 0)$ and $s \in (0, 1)$. It follows that $\bar{x} = 1$ is globally asymptotically stable with respect to $[0, \infty)$. We note that it would be just as simple to demonstrate the asymptotic stability using Theorem 2.1.2(b).

For continuous maps of the line, Theorem 2.1.2 settles the question of asymptotic stability conclusively, although it does not explicitly address instability, neutral stability, etc. A few extra paragraphs are needed to demonstrate the full reach of the preceding results and their applications. We start with a definition.