Section 2.2C: Notes.

Theorem 2.2.1 was first proved in Sharkovski (1964). An English translation of this original work in Russian has recently appeared in Sharkovski (1995). Presenting a proof of this theorem that is both readable *and* complete is somewhat difficult; see the comments in Misiurewicz (1997). The geometric style proof that is given here focuses on clarity rather than economy of expression; it is therefore long but hopefully readable and informative. Further, the presentation here is based on several excellent sources and thus benfits from their insights. These sources include Block, et al. (1980), Devaney (1989), Block and Coppel (1992), and Sharkovski, et al. (1993). These soruces in turn combine earlier works by Štefan (1977), Straffin (1978), Block, et al. (1980), Ho and Morris (1981), and Burkart (1982). A somewhat more algebraic proof of the theorem that is based on symbolic dynamics is given in Collet and Eckmann (1980), p.78.

In his 1964 paper, Sharkovski also proves a *converse* to Theorem 2.2.1, i.e., given a positive integer m, there is a continuous function $f : I \to I$ (e.g., $I = [0, 1]$) such that f has a cycle with period m but no cycles with period n if $n \rhd m$. The desired f can be constructed in a piecewise manner. In particular, one can construct a mapping f that has cycles with lengths 2^n for all integers $n \geq 0$ but no other cycles (Sharkovski, 1965). For the details and some related issues, see Coppel (1983), Block and Coppel (1992), Sharkovski, et al. (1993), Alsedà, et al. (1993), Sharkovski (1995) or Elaydi (1996). Some relevant historical remarks and comments pertaining to Theorem 2.2.1 and its converse appear in Misiurewicz (1997).

Corollary 2.2.1 was proved differently (and independently of Theorem 2.2.1) in Li and Yorke (1975). The observation cited in the first remark after the proof of Theorem 2.2.1 is from Block and Coppel (1992), p.12.

Like the major theorems of Section 2.1, Theorem 2.2.1 is not true if f is not continuous, or if its domain and range are not contained in the real line. For instance, consider the continuous map

$$f(e^{i\theta}) = e^{i(\theta + 2\pi/3)}$$

of the unit circle, which just the rotates the circle by 120 degrees, or 1/3 of the way. Obviously, every $\theta_0 \in [0, 2\pi)$ is a periodic point of period 3; yet, it is easy to see that f has periodic points of no other periods. However, *an analog of this theorem does hold for the circle*; specifically, continuous maps of the circle may be linked - through the standard lift map - to maps on the real line, where Theorem 2.2.1 applies; for details, refer to Block, et al. (1980) and Block and Coppel (1992).

Much of the material in Segment B is from Singer (1978), including the Schwarzian formula in Definition 2.2.2 (a slightly different formula appears in the nearly simultaneous article Allwright (1978), which has nearly the same content, though with a different focus). Also see Collet and Eckmann (1980), Section II.4, from which we have extracted Example 2.2.4. This latter example seemed slightly preferable to the quartic polynomial used in Singer (1978) or the exponential function in Allwright (1978); both of those examples essentially convey the same information as Example 2.2.4. A positive Schwarzian in a neighborhood of a fixed or periodic point can lead to unexpected bifurcations; see Section 2.3. Example 2.2.2 is Proposition 11.2 in Devaney (1989).