Section 4.3A: Weak contractions and stability.

We are interested here in maps f that satisfy the inequality

$$|f(X) - \bar{x}| < ||X - \bar{X}||, \qquad X \in D \subset \mathbb{R}^m. \tag{4.3b}$$

In the case $m = 1$, this reduces to the contraction inequality

$$|f(x) - \bar{x}| < |x - \bar{x}| \tag{4.3c}$$

which we already know implies asymptotic stability for $\bar{x}$, if (4.3c) holds in some neighborhood U of $\bar{x}$. In fact, (4.3c) generalizes linearization on the line since it implies the inequality $|f'(\bar{x})| \leq 1$ when f is C^1-smooth at $\bar{x}$. If the reverse of (4.3c) holds on U then $\bar{x}$ is strongly unstable (or repelling). Special cases of (4.3b) have appeared frequently in the published literature; in this section we encounter several instances of this occurrence in both theory and applications (also see the Notes segment below). To examine the consequences of (4.3b) when $m \geq 2$ we start with a definition.

Definition 4.3.1. Let Y be a fixed point of a continuous function F on $\mathbb{R}^m$. We say that F is a *weak contraction at a point X relative to Y* if

$$\|F(X) - Y\| \leq \|X - Y\|, \tag{4.3d}$$

and denote by $W(F, Y)$ the (closed) set of all points for which (4.3d) is satisfied. Clearly, $W(F, Y)$ is norm dependent, but we only use the sup-norm here. For a scalar map f, we call f a *weak contraction relative to a fixed point $\bar{x}$* if V_f is a weak contraction relative to $\bar{X}$. For f we also define the set

$$A(f; \bar{x}) \doteq \left\{X \in \mathbb{R}^m : |f(X) - \bar{x}| < \left\|X - \bar{X}\right\|\right\} \cup \{\bar{X}\}.$$

Next, some basic relationships are listed in a lemma (also see the Remark following Theorem 4.3.2). The simple proof is left to the reader.

Lemma 4.3.3. *Let $\bar{x}$ be a fixed point of f as defined by* (**??**). *Then*:

(a) $|f(X) - \bar{x}| \leq \left\|V_f(X) - \bar{X}\right\|$ *for all* X;

(b) *If x^* is a fixed point of* (**??**), *$x^* \neq \bar{x}$, then $x^* \in W(V_f, \bar{x})$ but not in $A(f; \bar{x})$*;

(c) $W(V_f, \bar{X}) = \{X : |f(X) - \bar{x}| \leq \left\|X - \bar{X}\right\|\} = \overline{A(f; \bar{x})}$.

(d) *$A(f; \bar{x})$ is open if and only if $\bar{X}$ is in the interior of $A(f; \bar{x})$.*

Note that if f is a weak contraction, then V_f is generally not a contraction (e.g., as in scalar linear maps - see Corollary 4.3.7 and the remarks preceding it); this fact has important consequences that will be discussed later on.

Lemma 4.3.3(b) shows that points in $W(V_f, \bar{X})$ that lie on the boundary of $A(f; \bar{x})$ may not be attracted to $\bar{X}$ under the iterations of V_f. However, the next result, and Corollary 4.3.3 show that points in $A(f; \bar{x})$ itself are different.

Theorem 4.3.1. (Asymptotic stability) *Let $\bar{x} \in \mathbb{R}$ be a fixed point of* (**??**) *and let T be a closed, invariant set containing $\bar{X}$, and let $A \doteq A(f; \bar{x})$. Then $\bar{X}$ is* *asymptotically stable relative to each invariant subset S of $T \cap A$ that is closed in T; in particular, $\bar{x}$ attracts every trajectory with a vector of initial values $(x_{1-m}, \ldots, x_0) \in S$.*

Proof. Let S be a T-closed, invariant subset of $T \cap A$. Then

$$\left|f(V_f^k(X)) - \bar{x}\right| < \left\|V_f^k(X) - \bar{X}\right\|. \tag{T4.3.1a}$$

for all positive integers k and for every $X \in S$. Next, for $1 \leq k \leq m-1$, observe that by (**??**),

$$V_f^k(X) = (\, f(V_f^{k-1}(X)), \ldots,\, f(X), u_1, \ldots, u_{m-k}) \tag{T4.3.1b}$$

for $X = (u_1, \ldots, u_m) \in A$. Now, by (T4.3.1b), Lemma 4.3.3(a) and induction on k,

$$\left\|V_f^k(X) - \bar{X}\right\| \leq \left\|X - \bar{X}\right\| \tag{T4.3.1c}$$

for $k = 1, \ldots, m-1$. Therefore, from (T4.3.1a), (T4.3.1c) and (**??**) we may conclude that

$$\begin{aligned}\left\|V_f^m(X) - \bar{X}\right\| &= \max\left\{\left|f(V_f^{m-1}(X)) - \bar{x}\right|, \ldots, |f(X) - \bar{x}|\right\} \\ &< \left\|X - \bar{X}\right\| \end{aligned} \tag{T4.3.1d}$$

for all $X \in S$. Next, let $X_0 = (x_0, \ldots, x_{1-m})$ be any vector of initial values for (**??**) in S. Then (T4.3.1d) implies that

$$\begin{aligned}\left\|V_f^{mn}(X_0) - \bar{X}\right\| &= \left\|V_f^m(V_f^{m(n-1)}(X_0)) - \bar{X}\right\| \\ &< \left\|V_f^{m(n-1)}(X_0) - \bar{X}\right\|\end{aligned}$$

for every positive integer n. Therefore, the sequence

$$\left\{\left\|V_f^{mn}(X_0) - \bar{X}\right\|\right\}, \qquad n = 1, 2, 3, \ldots$$

is strictly decreasing to a limit $r_0 \geq 0$ as $n \to \infty$. If $r_0 > 0$ and Ω_0 is the (forward) limit set of the vector sequence $\{V_f^{mn}(X_0)\}$, then

$$\Omega_0 \subset \partial B_{r_0}(\bar{X}) \cap S \subset T \cap A$$

where the first inclusion holds because S is closed in T. Therefore, by the invariance of Ω_0 under V_f^m, for any $Y \in \Omega_0$, (T4.3.1d) implies

$$r_0 = \left\|V_f^m(Y) - \bar{X}\right\| < \left\|Y - \bar{X}\right\| = r_0$$

which is impossible. Hence, for every $X_0 \in S$, $V_f^{mn}(X_0) \to \bar{X}$ as $n \to \infty$. By Lemmas 4.3.1 and 4.3.2,

$$V_f^{mn}(X_0) = (x_{mn-1}, \ldots, x_{m(n-1)})$$

so it may be concluded that for all $i = 1, \ldots, m$,

$$|x_{mn-i} - \bar{x}| \leq \left\| V_f^{mn}(X_0) - \bar{X} \right\| \to 0$$

as $n \to \infty$. It follows that $x_n \to \bar{x}$; thus $\bar{X}$ attracts every point of S, and S is closed, so $\bar{X} \in S$. With Lemma 4.3.3(c) implying the stability of $\bar{X}$ in the relative topology on A, the proof is complete.

Theorem 4.3.1 is valid with $T = \mathbb{R}^m$; on the other hand, in applied models, usually the cone $T = [0, \infty)^m$ is the relevant invariant set, to which attention may be restricted. It should be emphasized that if $T \neq \mathbb{R}^m$, then it is possible that $\bar{X} \in \partial T$, as in the next example. Indeed, Theorem 4.3.1 tends to be most useful when applied to such boundary equilibria (but also see Corollary 4.3.3).

Example 4.3.1. The third order equation

$$\begin{aligned} &x_n = ax_{n-1} + bx_{n-3} \exp\left(-cx_{n-1} - dx_{n-3}\right), \\ &\quad a, b, c, d \geq 0, \ c + d > 0 \end{aligned} \tag{E4.3.1a}$$

represents a special case of the flour beetle population model; see the Notes segment. We show that the origin is asymptotically stable (i.e., the beetles go extinct) if

$$a + b \leq 1, \quad b > 0. \tag{E4.3.1b}$$

Before proceeding, it is worth noting that the linearization of (E4.3.1a) at the origin has a unit eigenvalue $\lambda = 1$ when $a + b = 1$. Now, observe that if (E4.3.1b) holds, then for all $(x, y, z) \in [0, \infty)^3$, $(x, y, z) \neq (0, 0, 0)$,

$$\begin{aligned} ax + bz \exp(-cx - dz) &\leq [a + b \exp(-cx - dz)] \max\{x, z\} \\ &< (a + b) \max\{x, y, z\} \\ &\leq \max\{x, y, z\} \end{aligned}$$

so Theorem 4.3.1 implies that the origin is a stable global attractor of non-negative solutions. In a later segment below, we also derive sufficient conditions for the global stability of the *positive* fixed point of (E4.3.1a).

For comparison, we also obtain conditions that imply the boundedness of all solutions of (E4.3.1a). We note that if $d > 0$ then for all $x, z \geq 0$,

$$ax + bze^{-cx-dz} \leq ax + \frac{bz}{e^{dz}} \leq ax + \frac{b}{de}$$

so by Theorem 4.2.1, (E4.3.1a) is semipermanent if $0 \leq a < 1$, regardless of the value of b. Further, for each positive solution $\{x_n\}$, Lemmas 4.2.2 and 4.2.3 imply that an upper bound is the fixed point of the linear mapping $ax + b/de$, namely, $b/[de(1 - a)]$.

A generalization of Example 4.3.1 is the next corollary, which in particular provides a tool for establishing the global stability of the zero equilibrium. The simple proof (showing that the map is a weak contraction) is omitted.

Corollary 4.3.1. *Let* $f_i \in C([0,\infty)^m, [0,1))$ *for* $i = 1, \ldots, k$ *and* $k \geq 2$. *If* $\sum_{i=1}^{k} f_i(u_1, \ldots, u_m) < 1$ *for all* $(u_1, \ldots, u_m) \in [0,\infty)^m$ *then the origin is the unique, globally asymptotically stable fixed point of the following equation*:

$$x_n = \sum_{i=1}^{k} f_i(x_{n-1}, \ldots, x_{n-m}) x_{n-i}.$$

The next result is an immediate consequence of Theorem 4.3.1, and may be compared with Corollary 4.2.2.

Corollary 4.3.2. *The origin is a globally asymptotically stable fixed point of the equation*

$$x_n = x_{n-k} g(x_{n-1}, \ldots, x_{n-m}), \quad 1 \leq k \leq m$$

where $g \in C(\mathbb{R}^m, \mathbb{R})$, *if* $|g(X)| < 1$ for all $X \neq (0, \ldots, 0)$.

As a simple application of Corollary 4.3.2, note that the equation

$$x_n = a x_{n-k} \exp\left[-b(x_{n-1}^2 + \cdots + x_{n-m}^2)\right], \qquad |a| \leq 1, b > 0$$

has a globally asymptotically stable fixed point at the origin. When $k = m$ and $|a| = 1$, then the characteristic polynomial of this equation is $\lambda^m - a = 0$. Therefore, *every* eigenvalue lies on the unit circle in this case. For $|a| < 1$ the global nature of asymptotic stability cannot be inferred from linearization alone.

Corollary 4.3.3. (Asymptotic stability) *Let* A *be open in Theorem 4.3.1. Then* $\bar{x}$ *is asymptotically stable relative to* $(\bar{x} - r, \bar{x} + r)$, *where* $r > 0$ *is the largest real number such that* $B_r(\bar{X}) \subset A$. *In particular, if* $A = \mathbb{R}^m$, *then* $\bar{x}$ *is globally asymptotically stable.*

Proof. Although the ball in the statement of the corollary is not closed, note that for r_0 in the proof of Theorem 4.3.1, $\partial B_{r_0}(\bar{X}) \subset B_r(\bar{X})$ if $X_0 \in B_r(\bar{X})$. Hence, $\Omega_0 \subset A$ and the rest of the proof of Theorem 4.3.1 is applicable.

Remark. Geometrically, Corollary 4.3.3 states that if the graph of f in $\mathbb{R}^{m+1}$ lies within the *complement* of the closed polyhedral cone

$$C_{m+1}(\bar{x}) \doteq \{(u_1, \ldots, u_{m+1}) : |u_{m+1} - \bar{x}| \geq \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}\}$$

for all $(u_1, \ldots, u_m)$ in a deleted neighborhood of $\bar{X}$, then A is open and $\bar{x}$ is attracting as in Corollary 4.3.3. See Figure 4.3.1.