Section 5.3C: Chaotic competition and exclusion.

Chaotic behavior may also occur if the α_i are *substantially* different from each other, although Theorem 5.3.1 is no longer applicable. Furthermore, chaotic behavior occurs even when the matrix C does not satisfy the conditions of Theorem 5.3.1, and indeed, even when snap-back repellers do not exist. To site a case that complements that in Theorem 5.3.1, suppose that all rows of the matrix C are identical, i.e.,

$$c_{ij} = c_i > 0, i, j = 1, \ldots, m-1. \tag{5.3b}$$

In this case, demand is attenuated by the same factor for each good i (except possibly the m-th good) namely, $\exp\left[-\sum_{j=1}^{m-1} c_j x_j\right]$ and F takes the form

$$F(x_1, \ldots, x_{m-1}) = e^{-c_1 x_1 - \cdots - c_{m-1} x_{m-1}} [e^{\alpha_1} x_1, \ldots, e^{\alpha_{m-1}} x_{m-1}].$$

This is economically feasible if the first $m-1$ goods are similar to, and can be substituted for, each other. In particular, such comparable goods may compete for the consumer's attention through prices and other means; see the Remarks following the next theorem. We now give a complete description of the dynamics of system (**??**) under conditions (5.3b). In Section 6.2, an alternative but similar form of the following theorem is given in the context of a political science model.

Theorem 5.3.2. *Assume that C satisfies* (5.3b). *Then the following are true:*

(a) *The map F* is *$\mathbb{R}$-semiconjugate on $(0,\infty)^{m-1}$ to a linear mapping $\phi(t) = \omega t$ with $\omega \geq 1$ and $t > 0$. If $\omega > 1$, then each trajectory $\{F^n(x_0)\}$ approaches a subspace of $(0,\infty)^{m-1}$ obtained by setting one of the coordinates equal to zero.*

(b) *If none of the α_i are equal, and α_k is the largest among them, then $\{F^n(x_0)\}$ approaches a subset of the positive k-th axis. In particular, if the map $h(t) \doteq t\exp(\alpha_k - c_k t)$ is chaotic, then all positive trajectories $\{F^n(x_0)\}$ converge to a chaotic attractor on the positive k-th axis.*

(c) *If $\alpha_i = \alpha > 0$ for $i = 1, \ldots, m-1$, then F is radial and $\mathbb{R}$-semiconjugate to the function g in Lemma* 5.3.1. *Further, for each vector x_0 of initial values, the restriction of F to the ray $R_{x_0} = \{rx_0 : r \geq 0\}$ is topologically conjugate to g. In particular, if α is large enough that g is chaotic (e.g., $\alpha \geq 3.13$) then F is chaotic.*

Proof. (a) We may suppose without loss of generality that α_1 is the least among α_i. Then, in particular, $(m-2)\alpha_1 \leq \alpha_2 + \cdots + \alpha_{m-1}$. Define

$$H(x_1, \ldots, x_{m-1}) \doteq \frac{x_2 x_3 \cdots x_{m-1}}{x_1^{m-2}}, \quad \omega \doteq e^{\alpha_2 + \cdots + \alpha_{m-1} - (m-2)\alpha_1}.$$

Note that $\omega \geq 1$. Semiconjugacy to ϕ with H as link is readily verified, since

$$\begin{aligned} H(F(x_1, \ldots, x_{m-1})) &= \frac{e^{\alpha_2 + \cdots + \alpha_{m-1}}}{e^{(m-2)\alpha_1}} \frac{x_2 x_3 \cdots x_{m-1}}{x_1^{m-2}} \\ &= \omega H(x_1, \ldots, x_{m-1}). \end{aligned}$$

Let x_0 be any point in $(0,\infty)^{m-1}$. Since all c_i are positive, F is bounded on $(0,\infty)^{m-1}$. In particular, there is $0 < \mu \leq b$ such that each component $F_i(x_i) \leq \mu$ for $x_i > 0$. It follows that the trajectory $\{F^n(x_0)\}$ is in $(0,\mu]^{m-1}$ for all $n \geq 1$. Writing $F^n(x_0) = (x_{1,n}, \ldots, x_{m-1,n})$, we note that

$$\begin{aligned}\frac{x_{2,n}x_{3,n}\cdots x_{m-1,n}}{x_{1,n}^{m-2}} &= H(F^n(x_0)) \\ &= \phi^n(H(x_0)) \\ &= \frac{x_{2,0}x_{3,0}\cdots x_{m-1,0}}{x_{1,0}^{m-2}}\,\omega^n.\end{aligned}$$

If $\omega > 1$, then $H(F^n(x_0)) \to \infty$ as $n \to \infty$, although the product $x_{2,n}x_{3,n}\cdots x_{m-1,n} \leq \mu^{m-2}$ is bounded. It follows that $x_{1,n} \to 0$ as $n \to \infty$ and therefore, $\{F^n(x_0)\}$ approaches the subspace $x_1 = 0$, as claimed in the statement of the theorem.

(b) We may suppose that $0 < \alpha_1 < \cdots < \alpha_{m-1}$ (so $k = m-1$). By applying Part (a) repeatedly to maps

$$e^{-c_jx_j-\cdots-c_{m-1}x_{m-1}}[0,\ldots,0,e^{\alpha_j}x_j,\ldots,e^{\alpha_{m-1}}x_{m-1}]$$

where $1 \leq j \leq m-1$, we observe that the only coordinate that does not vanish asymptotically is $m-1$. Further, the mapping $e^{-c_{m-1}x_{m-1}}[0,\ldots,0,\,e^{\alpha_{m-1}}x_{m-1}]$ is topologically conjugate to the map h. So if h is chaotic, then $\{F^n(x_0)\}$ approaches the chaotic attractor of h.

(c) In this case, F takes the following form:

$$F(x_1,\ldots,x_{m-1}) = \exp\left(\alpha - \sum_{j=1}^{m-1} c_jx_j\right)[x_1,\ldots,x_{m-1}]$$

which is obviously radial. Defining $H(x_1,\ldots,x_{m-1}) \doteq \sum_{j=1}^{m-1} c_jx_j$, it is easy to see that

$$\begin{aligned}H(F(x_1,\ldots,x_{m-1})) &= H(x_1,\ldots,x_{m-1})e^{\alpha - H(x_1,\ldots,x_{m-1})} \\ &= g(H(x_1,\ldots,x_{m-1}))\end{aligned}$$

which shows F on $[0,\infty)^{m-1}$ to be $\mathbb{R}$-semiconjugate to g on $[0,\infty)$. The H-fibers are the parts of hyperplanes

$$\sum_{j=1}^{m-1} c_jx_j = t \geq 0$$

that are contained in the cone $[0,\infty)^{m-1}$ (clearly, all such fibers are compact). The rest of the proof is now clear, since as in the proof of Lemma 5.3.1, for each x_0 the ray R_{x_0} is homeomorphic to $[0,\infty)$ and the restriction of F to R_{x_0} is topologically conjugate to g of Lemma 5.3.1.

Remarks.

1. (Structural stability) From Theorem 5.3.2 we see that the behavior in Part (c) where all α_i are equal is not structurally stable. In particular, there can be no snap-back repellers in the semiconjugate case. However, when strict inequality holds among the α_i, then the semiconjugate case is obviously structurally stable.

2. (Competition among similar goods) We argued above that (5.3b) is economically feasible if the first $m-1$ goods are similar enough to be substitutable for each other. In such a case, the consumer may choose one among them and ignore the rest. *According to Theorem 5.3.2, the consumer chooses the good with the largest α_i value* (in the exceptional case that two or more α_i have the same highest value, then the consumer chooses a mix of these latter goods, with the proportions in the mix arbitrarily determined by x_0).

Recall that $\alpha_i = \ln(b/p_i) + \beta_i + \gamma_{im}b$. Thus $\alpha_i < \alpha_j$ if and only if

$$\ln\left(\frac{p_j}{p_i}\right) < \tau_j - \tau_i, \quad \tau_k \doteq \beta_k + \gamma_{km}b. \tag{5.3c}$$

In particular, if goods i and j are viewed equally by the consumer (e.g., neither is a brand name or particularly preferred for some reason), then $\tau_j - \tau_i = 0$, so (5.3c) implies that $p_j < p_i$. Thus, as might be expected, the consumer buys the lower priced good when all else is equal.

If we are interested *only in which goods will be eliminated* (rather than the chaotic nature or other details of the asymptotic demand behavior) then conditions (5.3b) can be relaxed. The following gives a possible extension.

Theorem 5.3.3. *If for the l-th good, it is true that*

$$\sum_{\substack{i=1\\ i\neq l}}^{m-1} \frac{c_{ij}}{a_i} < (m-2)\frac{c_{lj}}{a_l}, \quad j = 1, \ldots, m-1, \tag{T5.3.3}$$

then $\{x_{l,n}\}$ converges to zero; i.e., the l-th good is eliminated.

Proof. Define

$$z_n \doteq \frac{x_l^{(m-2)/a_k}}{\prod_{\substack{i=1\\ i\neq l}}^{m-1} x_i^{1/a_i}}$$

and for each $n = 0, 1, 2, \ldots$ note that

$$\frac{x_{l,n+1}^{(m-2)/a_l}}{\prod_{\substack{i=1\\ i\neq l}}^{m-1} x_{i,n+1}^{1/a_i}} = \frac{x_{l,n}^{(m-2)/a_l}}{\prod_{\substack{i=1\\ i\neq l}}^{m-1} x_{i,n}^{1/a_i}} \exp\left(\sum_{\substack{i=1\\ i\neq l}}^{m-1}\sum_{j=1}^{m-1} \frac{c_{ij}}{a_i}x_{j,n} - (m-2)\sum_{j=1}^{m-1}\frac{c_{lj}}{a_l}x_{j,n}\right)$$

Upon switching the double sum's order, (T5.3.3) implies that $z_{n+1} < z_n$ for all n. If $\lim_{n\to\infty} z_n = z$, then $z \geq 0$. If $z = 0$, then the boundedness of the denominator in z_n implies that $x_{l,n} \to 0$ as $n \to \infty$. If $z > 0$, then let

$$\limsup_{n\to\infty} x_{i,n} = \lim_{k_i\to\infty} x_{i,n_{k_i}} = \sigma_i, \quad i = 1, \ldots, m-1,$$

and note that $\sigma_i < \infty$ for each i. Hence,

$$\begin{aligned} z_{n_{k_i}+1} &= z_{n_{k_i}} \exp\left(\sum_{\substack{i=1 \\ i\neq l}}^{m-1} \sum_{j=1}^{m-1} \frac{c_{ij}}{a_i} x_{j,n_{k_i}} - (m-2) \sum_{j=1}^{m-1} \frac{c_{lj}}{a_l} x_{j,n_{k_i}} \right) \\ &\leq z_{n_{k_i}} \exp\left[\sum_{j=1}^{m-1} \left(\sum_{\substack{i=1 \\ i\neq l}}^{m-1} \frac{c_{ij}}{a_i} - (m-2)\frac{c_{lj}}{a_l} \right) \sigma_j \right] \end{aligned}$$

and taking the limit as $k_i \to \infty$ we obtain

$$z \leq z \exp\left[\sum_{j=1}^{m-1} \left(\sum_{\substack{i=1 \\ i\neq l}}^{m-1} \frac{c_{ij}}{a_i} - (m-2)\frac{c_{lj}}{a_l} \right) \sigma_j \right]$$

so that $\sigma_j = 0$ for every j. In particular, $x_{l,n}$ converges to zero.