Section 5.6A: The model (of combat)

Our focus here is on the ground component of Epstein's model which involves the following non-negative quantities:

A_n is the attacker's combat power (or lethality) in period (e.g., day) n;
D_n is the defender's combat power (or lethality) in period n;
α_n is the attrition rate of the attacker in period n;
δ_n is the attrition rate of the defender in period n;
$a \in (0,1)$ is the attacker's prescribed attrition rate threshold;
$d \in (0,1)$ is the defender's prescribed attrition rate threshold;
W_n is the defender's rate of withdrawal in period n;
$W_{\max}$ is the defender's prescribed maximum rate of withdrawal;
γ_n is the attacker's "prosecution rate" of combat in period n.

See the Notes segment and Example 5.6.1 below for more information. The following relationships exist by definition between the various quantities defined above:

$$W_n \leq W_{\max} \quad \text{for all } n, \quad W_0 = 0,\ W_{\max} > 0$$

$$A_{n+1} = A_n - \alpha_n A_n = (1 - \alpha_n) A_n,\ A_0 > 0 \tag{5.6a}$$

$$D_{n+1} = D_n - \delta_n D_n = (1 - \delta_n) D_n,\ D_0 > 0 \tag{5.6b}$$

Following Epstein, we further postulate the relationships:

$$\alpha_n = \gamma_n (1 - W_n / W_{\max}), \quad \alpha_0, \gamma_0 \in (0, a], \tag{5.6c}$$

$$\gamma_{n+1} = \gamma_n + \frac{1}{a}(a - \gamma_n)(a - \alpha_n), \tag{5.6d}$$

$$W_{n+1} = \begin{cases} W_n + [(W_{\max} - W_n)/(1-d)](\delta_n - d), & \delta_n \geq d \\ 0, & \delta_n < d \end{cases} \tag{5.6e}$$

Equation (5.6e), which is of particular interest to us here, incorporates the defender's decision process: He will withdraw by the indicated amount only when his attrition rate δ_n reaches the threshold value d. Since the value of W_n may not be zero when $\delta_n = d$, (5.6e) clearly injects a discontinuity into the system.

Next, we define the following "exchange ratio" of attacker units lost per defender units lost in day n:

$$\rho_n = \frac{\alpha_n A_n}{\delta_n D_n}. \tag{5.6f}$$

Like Epstein, for simplicity we take ρ_n to be constant, say, $\rho_n = \rho$ (also see the Notes section below). Let us now define the following three non-negative *pure rates* variables:

$$x_n = \gamma_n, \quad y_n = \frac{W_n}{W_{\max}}, \quad z_n = \frac{\delta_n}{\alpha_n}.$$

From Equations (5.6a) and (5.6b) we obtain:

$$\frac{A_{n+1}}{D_{n+1}} = \frac{(1-\alpha_n)A_n}{(1-\delta_n)D_n}$$

which is transformed, using (5.6f) with constant ρ, into

$$\frac{\delta_{n+1}}{\alpha_{n+1}} = \frac{(1-\alpha_n)}{(1-\delta_n)}\frac{\delta_n}{\alpha_n}. \tag{5.6g}$$

Given that $\delta_n = \alpha_n z_n$, (5.6g) may be written as

$$z_{n+1} = \frac{1-\alpha_n}{1-\alpha_n z_n} z_n.$$

Transforming the rest of the preceding equations into x, y, z, we obtain the dynamical system:

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{a}(a - x_n)[a - x_n(1-y_n)] \\ y_{n+1} &= \begin{cases} y_n + \frac{1-y_n}{1-d}[z_n x_n(1-y_n) - d], & z_n x_n(1-y_n) \geq d \\ 0, & z_n x_n(1-y_n) < d \end{cases} \\ z_{n+1} &= \frac{1 - x_n(1-y_n)}{1 - z_n x_n(1-y_n)} z_n \end{aligned} \tag{5.6h}$$

with initial values given as follows:

$$x_0 \in (0, a],\ y_0 = 0,\ z_0 = \frac{A_0}{\rho D_0}. \tag{5.6i}$$

The dynamical system consisting of (5.6h) and (5.6i) constitute Epstein's model without air support. Clearly, if x_n, y_n and z_n are known, then the remaining variables are easily determined from the remaining (passive) equations:

$$\alpha_n = x_n(1-y_n), \quad \delta_n = z_n x_n(1-y_n)$$

and

$$A_n = A_0 \prod_{i=0}^{n-1}[1 - x_i(1-y_i)], \quad D_n = \frac{A_n}{\rho z_n}. \tag{5.6j}$$

It is evident from the second equation in (5.6h) that we have a bimodal system. The two modes are (F_i, D_i), $i = 1, 2$, where

$$\begin{aligned} D_1 &= \{(x, y, z) :\ zx(1-y) < d]\} \\ D_2 &= \{(x, y, z) :\ zx(1-y) \geq d\} \end{aligned}$$

and

$$\begin{aligned} F_1 &= [f_1(x, y, z), 0, f_3(x, y, z)], \\ F_2 &= [f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)] \end{aligned}$$

where

$$f_1(x,y,z) \doteq x + \frac{1}{a}(a-x)[a-x(1-y)]$$
$$f_2(x,y,z) \doteq y + \frac{1-y}{1-d}[zx(1-y)-d]$$
$$f_3(x,y,z) \doteq \frac{1-x(1-y)}{1-zx(1-y)}z.$$

We are primarily concerned with the cube $[0,1]^3$ in so for as the combat model is concerned, since x, y, z represent rates in that model. The cube is not generally invariant under the collective actions of the F_i, but as we will see below, under suitable restrictions, trajectories starting from certain initial points do remain in $[0,1]^3$. The common boundary between D_1 and D_2 in the cube may be represented as part of the surface whose equation is given by the function

$$z = \frac{d}{x(1-y)}.$$

We call the mode (F_1, D_1) the *engagement mode* (since no withdrawal takes place) and call (F_2, D_2) the *withdrawal mode*.