# Spherical Harmonics

## David Park, djmp@earthlink.net

Exercise [22.32] in *The Road To Reality*, Roger Penrose, 2004, Alfred A. Knopf, New York, concerns calculating the Laplacian operator in spherical coordinates. He classifies this as an exercise 'not to be undertaken lightly' - unless you have *Tensorial*.

## Initialization

```
In[1]:=  Needs["TensorCalculus4`Tensorial`"]
```

We declare base indices for spherical coordinates on the sphere. Just for fun we will use red indices. We define tensor shortcuts for `x` the coordinate position, the metric matrix `g`, the Kronecker `δ`, the Christoffel symbols `Γ`.

```
In[2]:=  DeclareBaseIndices[{θ, ϕ}]
         DeclareIndexFlavor[{red, Red}]
         DefineTensorShortcuts[{{x}, 1}, {{g, δ}, 2}, {{Γ}, 3}]
         labs = {x, δ, g, Γ};
```

## Metric for the Surface of a Sphere

Define and set metric tensor for the surface of a sphere.

```
In[6]:=  (cmetric = DiagonalMatrix[{1, Sin[θ]^2}]) // MatrixForm
         metric = cmetric // CoordinatesToTensors[{θ, ϕ}, x, red];
         MapThread[SetTensorValueRules[#1, #2] &,
           {{gdd[a, b], guu[a, b]} // ToFlavor[red], {metric, Inverse@metric}}];
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Calculate and set the Christoffel symbols.

```
In[9]:=  MapThread[SetTensorValueRules[#1, #2] &,
           {{Γddd[a, b, c], Γudd[a, b, c]} // ToFlavor[red], CalculateChristoffels[labs, red]}];
         SelectedTensorRules[Γ, Γudd[_, b_, c_] /; OrderedQ[{b, c}]] //
           UseCoordinates[{θ, ϕ}, x, red] // TableForm
```

Out[10]//TableForm=

$$\Gamma^{\theta}{}_{\phi\,\phi} \to -\mathrm{Cos}[\theta]\ \mathrm{Sin}[\theta]$$

$$\Gamma^{\phi}{}_{\theta\,\phi} \to \mathrm{Cot}[\theta]$$

The last statement displayed the nonzero, independent up Christoffel values.

## Calculate the $\nabla^2$ Operator

```
In[11]:=  SetCovariantDisplay["DelMode"]
```

We calculate the Laplacian operating on a function, starting with the general definition.

```
In[12]:=  Print["General definition of the Laplacian."]
          guu[a, b] CovariantD[Tensor[ψ], {a, b}] // ToFlavor[red]
          Print["Expanded in terms of the coordinate positions."]
          %% // ExpandCovariantD[labs, red /@ {c, d}]
          Print[
           "Performing the summations and substituting values for spherical coordinates."]
          %% // ToArrayValues[]
          Print["Converting to coordinate symbols"]
          step1 = %% // UseCoordinates[{θ, ϕ}, x, red]
```

```
General definition of the Laplacian.
```

Out[13]= $\nabla_{a\,b}\psi\ g^{a\,b}$

```
Expanded in terms of the coordinate positions.
```

Out[15]= $g^{a\,b}\left(\dfrac{\partial^2\psi}{\partial x^{a}\,\partial x^{b}} - \Gamma^{d}{}_{b\,a}\,\dfrac{\partial\psi}{\partial x^{d}}\right)$

```
Performing the summations and substituting values for spherical coordinates.
```

Out[17]= $\dfrac{\partial^2\psi}{\partial x^{\theta}\,\partial x^{\theta}} + \mathrm{Csc}[x^{\theta}]^2\,\dfrac{\partial^2\psi}{\partial x^{\phi}\,\partial x^{\phi}} + \mathrm{Cot}[x^{\theta}]\,\dfrac{\partial\psi}{\partial x^{\theta}}$

```
Converting to coordinate symbols
```

Out[19]= $\mathrm{Cot}[\theta]\,\dfrac{\partial\psi}{\partial\theta} + \dfrac{\partial^2\psi}{\partial\theta\,\partial\theta} + \mathrm{Csc}[\theta]^2\,\dfrac{\partial^2\psi}{\partial\phi\,\partial\phi}$

That is the answer that Penrose has. We can convert it to textbook 'operator' form by...

```
In[20]:= step1 /. ψ → ""
```

*Out[20]=*

$$\mathrm{Cot}[\theta]\ \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta\,\partial\theta} + \mathrm{Csc}[\theta]^2\ \frac{\partial^2}{\partial\phi\,\partial\phi}$$

We could define an active spherical Laplacian operator as follows, or we could use the method in the second cell below.

```
In[21]:= spherelaplacian[θ_, ϕ_] :=
          Function[ψ, Cot[θ] D[ψ, θ] + D[ψ, {θ, 2}] + Csc[θ]^2 D[ψ, {ϕ, 2}]]
         spherelaplacian[θ, ϕ][ψ[θ, ϕ]]
```

*Out[22]=*

$$\mathrm{Csc}[\theta]^2\ \psi^{(0,2)}[\theta, \phi] + \mathrm{Cot}[\theta]\ \psi^{(1,0)}[\theta, \phi] + \psi^{(2,0)}[\theta, \phi]$$

The spherical Laplacian corresponds to the operator for total angular momentum. We can write the corresponding eigenvalue equation as follows.

```
In[23]:= Print["Eigenvalue equation for spherical harmonics"]
         step1 == k Tensor[ψ]
         Print["Converting to regular Mathematica differential equation"]
         pdeqn = %% /. Tensor[ψ] → ψ[θ, ϕ]
```

```
Eigenvalue equation for spherical harmonics
```

*Out[24]=*

$$\mathrm{Cot}[\theta]\ \frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta\,\partial\theta} + \mathrm{Csc}[\theta]^2\ \frac{\partial^2\psi}{\partial\phi\,\partial\phi} == k\,\psi$$

```
Converting to regular Mathematica differential equation
```

*Out[26]=*

$$\mathrm{Csc}[\theta]^2\ \psi^{(0,2)}[\theta, \phi] + \mathrm{Cot}[\theta]\ \psi^{(1,0)}[\theta, \phi] + \psi^{(2,0)}[\theta, \phi] == k\,\psi[\theta, \phi]$$

The solution of this equation leads to the spherical harmonics but since it is outside the exercise and outside tensor calculus we will stop here.