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Adaptive Classification of Temporal Signals in Fixed-Weight Recurrent Neural Networks: An Existence Proof

Ivan Y. Tyukin
I.Tyukin@le.ac.uk
Department of Mathematics, School of Mathematics and Computer Science, University of Leicester, Leicester, LE1 7RH, U.K., and
Laboratory for Perceptual Dynamics, Brain Science Institute, RIKEN, Wako-shi, Saitama, 351-0198, Japan

Danil Prokhorov
dvprokhorov@gmail.com
Toyota Technical Center, Ann Arbor, MI 48105, U.S.A.

Cees van Leeuwen
ceesvl@brain.riken.jp
Laboratory for Perceptual Dynamics, Brain Science Institute, RIKEN, Wako-shi, Saitama, 351-0198, Japan

Recurrent neural networks with fixed weights have been shown in practice to successfully classify adaptively signals that vary as a function of time in the presence of additive noise and parametric perturbations. We address the question: Can this ability be explained theoretically? We provide a mathematical proof that these networks have this ability even when parametric perturbations enter the signals nonlinearly. The restrictions that we impose on the signals to be classified are that they satisfy an assumption of nondegeneracy and that noise amplitude is sufficiently small. Further, we demonstrate that the recurrent neural networks may not only classify uncertain signals adaptively but also can recover the values of uncertain parameters of the signals, up to their equivalence classes.

1 Introduction

Recurrent neural networks (RNN) with fixed weights are known to solve problems of adaptive classification, recognition, and control (Prokhorov, Feldkamp, & Tyukin, 2002; Feldkamp, Puskorius, & Moore, 1996; Feldkamp & Puskorius, 1997; Younger, Conwell, & Cotter, 1999; Lo, 2001). When the objects to be classified are static, for example still images or vectors in $\mathbb{R}^n$, solutions to these problems are usually characterized in terms of convergence of the RNN state to an attractor (Hopfield, 1982, Fuchs & Haken, 1988a, 1988b).[1] Each attractor corresponds to a specific class of objects, and its basin determines which objects belong to the class. Conditions specifying convergence to an attractor are widely available in this case (Cohen & Grossberg, 1983; Michel, Farrel, & Porod, 1989; Yang, & Dillon, 1994; Chen & Amari, 2001; Lu & Chen, 2003).

When the objects to be classified are dynamic, for instance, nonlinearly parameterized functions of time of which the parameters are unknown a priori, no adequate theory exists to explain why the fixed-weight RNN approach is successful. At present, theoretical results are available to demonstrate that a single fixed-weight RNN of a certain type can approximate the solutions of multiple dynamical systems (Back & Chen, 2002). Hence, in principle, a fixed-weight RNN can behave adaptively with respect to changes of its input signals. These theoretical results, however, are restricted to the class of parameter replacement networks (Chen & Chen, 1995). The structure of these networks differs from that of the more commonly used recurrent multilayered perceptrons.

It remains an unresolved theoretical issue whether adaptive behavior is inherent to these other types of RNN. Although several authors have given plausibility arguments that RNNs with conventional multilayer architecture should also have this ability (Feldkamp & Puskorius, 1997; Prokhorov et al., 2002), a formal proof has yet to be presented.

In this letter, we consider adaptive behavior in fixed-weight RNNs with a view to their ability to classify temporal signals adaptively. We provide a formal proof that continuous-time RNNs with fixed weights can successfully classify and recognize nonlinear functions of time, which are allowed to be nonlinearly parameterized with unknown parameter values. The main idea behind our results consists of presenting a prototype dynamical system that solves the recognition problem. We then present a proof that an RNN with fixed weights can realize this system. We construct such a system using the concepts of relaxation times and weakly attracting sets (Milnor, 1985; Gorban, 2004),[2] as well as the tests for convergence to such

[1] Let $\mathbf{x}$ be the state of an RNN, and $\mathbf{x}(t, \mathbf{x}_0)$ be the trajectory generated by the RNN's state over time $t \in [0, \infty)$ passing through the point $\mathbf{x}_0$ at $t = t_0 \geq 0$. An *attractor* (more precisely, an attracting set) in this context is generally understood as a closed, invariant set $\mathcal{A}$ such that for some neighborhood $\mathcal{V}$ of $\mathcal{A}$, the following holds: (1) for all points $\mathbf{x}_0 \in \mathcal{V}$ states $\mathbf{x}(t, \mathbf{x}_0)$ remain in $\mathcal{V}$ for all $t \geq t_0$, and (2) they converge asymptotically to $\mathcal{A}$ with time: $\mathbf{x}(t, \mathbf{x}_0) \to \mathcal{A}$ as $t \to \infty$. For a more detailed overview and discussion of the notion of attractor, see Guckenheimer and Holmes (2002) and Broer, Dumortier, Stiren, and Takens (1991).

[2] A set $\mathcal{A}$ is *weakly attracting*, or a Milnor attracting set, iff (1) it is closed and invariant and (2) for some set $\mathcal{V}$ (not necessarily a neighborhood of $\mathcal{A}$) with strictly positive measure and for all $\mathbf{x}_0 \in \mathcal{V}$, the following holds: $\mathbf{x}(t, \mathbf{x}_0) \to \mathcal{A}$ as $t \to \infty$. The main difference between the conventional notion of an attracting set and that of a weakly attracting set is that the domain of attraction of the latter is not necessarily a neighborhood of $\mathcal{A}$. This property will be essential for theoretical justification of the main results of our letter.

sets obtained in our earlier work (Tyukin, Steur, Nijmeijer, & van Leeuwen, 2008). To show that our system can indeed be realized by an RNN with fixed weights, we employ classical results on function approximation by feedforward networks (Cybenko, 1989).

The current letter is organized as follows. Section 2 describes notational conventions. In section 3 we provide a mathematical statement of the problem, section 4 contains the main results, and section 5 concludes the letter.

2 Notational Preliminaries

The following notations are used in this letter:

- Symbol $\mathbb{R}$ defines the field of real numbers, and symbol $\mathbb{R}_{\geq c}$, $c \in \mathbb{R}$ stands for the following set $\mathbb{R}_{\geq c} = \{x \in \mathbb{R} | x \geq c\}$, and $\mathbb{R}_{>c} = \{x \in \mathbb{R} | x > c\}$.
- Symbol $\mathbb{R}^n$ stands for an n-dimensional linear space over the field of reals.
- $\mathcal{C}^k$ denotes the space of functions that are at least k times differentiable.
- Symbol $\mathcal{K}$ denotes the class of all strictly increasing functions $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\kappa(0) = 0$; symbol $\mathcal{K}_\infty$ denotes the class of all functions $\kappa \in \mathcal{K}$ such that $\lim_{s\to\infty} \kappa(s) = \infty$.
- Symbol $\oplus$ denotes concatenation of two vectors.
- The solution of a system of differential equations $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\theta}, \mathbf{u}(t))$, $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{f} \in \mathcal{C}^0$, $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\boldsymbol{\theta} \in \mathbb{R}^d$ passing through point $\mathbf{x}_0$ at $t = t_0$ will be denoted for $t \geq t_0$ as $\mathbf{x}(t, \mathbf{x}_0, t_0, \boldsymbol{\theta}, \mathbf{u})$, or simply as $\mathbf{x}(t, \mathbf{x}_0)$ or $\mathbf{x}(t)$ if it is clear from the context what the values of $\mathbf{x}_0, \boldsymbol{\theta}$ are and how the function $\mathbf{u}(t)$ is defined.
- By $L^n_\infty[t_0, T]$, $t_0 \geq 0$, $T \geq t_0$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty,[t_0,T]} = \operatorname{ess\,sup}\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$; $\|\mathbf{f}\|_{\infty,[t_0,T]}$ stands for the $L^n_\infty[t_0, T]$ norm of $\mathbf{f}(t)$.
- Let $\mathcal{A}$ be a set in $\mathbb{R}^n$ and $\|\cdot\|$ be the usual Euclidean norm in $\mathbb{R}^n$. By the symbol $\|\cdot\|_{\mathcal{A}}$ we denote the following induced norm:

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf_{\mathbf{q}\in\mathcal{A}}\{\|\mathbf{x} - \mathbf{q}\|\}.$$

In case x is a scalar and $\Delta \in \mathbb{R}_{>0}$, notation $\|x\|_\Delta$ stands for the following function:

$$\|x\|_\Delta = \begin{cases} |x| - \Delta, & |x| > \Delta \\ 0, & |x| \leq \Delta \end{cases}.$$

3 Problem Formulation

3.1 Signals to Be Classified. We consider the following set of signals,

$$\begin{aligned}&\mathcal{F} = \{f_i(\xi(t), \theta_i)\},\ i \in \{1, \ldots, N_f\},\\&\quad f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ f_i(\cdot, \cdot) \in \mathcal{C}^0,\\&\quad \xi : \mathbb{R}_{\geq 0} \to \mathbb{R},\ \xi(\cdot) \in \mathcal{C}^1 \cap L_\infty[0, \infty],\end{aligned} \tag{3.1}$$

where $\theta_i \in \Omega_\theta \subset \mathbb{R}$ are parameters of which the values are unknown a priori, $\Omega_\theta = [\theta_{\min}, \theta_{\max}]$ is a bounded interval, and $\xi(t)$ is a known and bounded function. Signals $f_i(\xi(t), \theta_i)$ constitute the set of variables chosen to represent the state of an object.

For the given functions $f_i(\xi(t), \theta_i)$ and $\xi(t)$, we say that θ_i is equivalent to θ_i' iff

$$f_i(\xi(t), \theta_i) = f_i(\xi(t), \theta_i')\ \forall\, t \in \mathbb{R}_{\geq 0}. \tag{3.2}$$

Hence, an equivalence class for $\theta_i \in \Omega_\theta$ can be defined as

$$E_i(\theta_i) = \{\theta_i' \in \mathbb{R} |\ f_i(\xi(t), \theta_i) = f_i(\xi(t), \theta_i')\ \forall\, t \in \mathbb{R}_{\geq 0}\}. \tag{3.3}$$

Equivalence classes 3.3 determine sets of indistinguishable parameterizations of the ith signal. It is natural, therefore, to restrict ourselves to the problem of recognizing signals 3.1 up to their equivalence classes.

With respect to the equivalence classes $E_i(\theta_i)$, we further assume that there is at least one point $\theta_0 \in \mathbb{R}$ such that

$$\|\theta_0\|_{E_i(\theta_i)} \geq \Delta_\theta \in \mathbb{R}_{>0}\ \forall\, \theta_i \in \Omega_\theta. \tag{3.4}$$

Requirement 3.4 is a technical assumption. It is satisfied in a wide range of practically relevant situations in which the union of $E_i(\theta_i)$ for all i and θ_i belongs to an interval of $\mathbb{R}$. The requirement allows us to exclude from consideration pathological cases in which almost all points in Ω_θ are indistinguishable in the sense of condition 3.2.

In our work, we consider the case in which the values of $f_i(\xi(t), \theta_i)$ may not be available for direct observation. We assume that instead of functions $f_i(\xi(t), \theta_i)$, we access variables $s_i(t, s_{i,0}, \theta_i, \eta_i(t))$, which are solutions to the following ordinary differential equation:

$$\begin{aligned}\dot{s}_i &= -\varphi_i(s_i) + f_i(\xi(t), \theta_i) + \eta_i(t)\\ s_i(t_0) &= s_{i,0},\, s_{i,0} \in \Omega_s \subset \mathbb{R}.\end{aligned} \tag{3.5}$$

In equation 3.5, the function $\eta_i : \mathbb{R}_{>0} \to \mathbb{R}$,

$$\eta_i(t) \in L_\infty[0, \infty], \ \|\eta_i(t)\|_{\infty,[0,\infty]} \leq \Delta_\eta \in \mathbb{R}_{\geq 0}, \tag{3.6}$$

corresponds to measurement noise. The value of Δ_η in equation 3.5 is supposed to be known, while the values of initial conditions $s_i(t_0)$ and functions $\varphi_i : \mathbb{R} \to \mathbb{R}, \ \varphi(\cdot) \in \mathcal{C}^1$ in equation 3.5 are assumed to be uncertain. We do, however, require that $\Omega_s = [s_{\min}, s_{\max}]$ is an interval and that the functions $\varphi_i(s_i)$ satisfy the following constraint:

$$\forall s_i \in \mathbb{R} \Rightarrow \ \varphi_{\min} \leq \frac{\partial \varphi_i(s_i)}{\partial s_i} \leq \varphi_{\max}, \ \varphi_{\min}, \varphi_{\max} \in \mathbb{R}_{>0}. \tag{3.7}$$

Condition 3.7 ensures that the dynamics of each variable $s_i(t, s_{i,0}, \theta_i, \eta_i(t))$ at $t \to \infty$ is uniquely determined in the absence of noise by $f_i(\xi(t), \theta_i)$, and the effects of initial conditions $s_{i,0}$ vanish with time asymptotically (Pavlov, Wouw, & Nijmeijer, 2006). In other words, solutions $s_i(t, s_{i,0}, \theta_i, 0)$ will asymptotically approach a function of time of which the shape is uniquely determined by $f_i(\xi(t), \theta_i)$.

The reason we consider signals 3.5 to 3.7 instead of $f_i(\xi(t), \theta_i)$ is that in many systems, artificial or natural, measured physical quantities, represented here by signals $f_i(\xi(t), \theta_i)$, are often unavailable. In the domain of neural computation and modeling of neural systems, this is the case when information about the stimuli is transferred from one node to another by dynamic synapses (Tsodyks, Pawelzik, & Markram, 1998). In robotics and motor control, intrinsic dynamics of sensors and moving mechanical parts often distort stimulus information $f_i(\xi(t), \theta_i)$. Equations 3.5 are routinely used in the literature on system identification and adaptive control (Cao, Annaswamy, & Kojic, 2003). Hence considering this class of signals will provide theoretical justification for expanding RNN-based classifiers into wider areas of neural computation. This is done in our study.

Notice also that solutions to the following differential equation,

$$\begin{aligned} \dot{s}_i &= -\tau(s_i - f_i(\xi(t), \theta_i)) + \eta_i(t), \ \tau \in \mathbb{R}_{>0}, \\ \eta_i(t) &= \frac{d}{dt}(f_i(\xi(t), \theta_i)), \end{aligned}$$

with initial conditions $s_i(t_0) = f_i(\xi(t_0), \theta_i)$ coincide with $f_i(\xi(t), \theta_i)$ for all $t \geq t_0$.[3] Therefore, in case the derivative of $\xi(t)$ is uniformly bounded in t, it is always possible to consider signals $f_i(\xi(t), \theta_i)$ as if they were generated

[3]This can be easily verified by taking the time derivative of $e_i(t) = s_i(t, s_{i,0}, \theta_i, \eta_i(t)) - f_i(\xi(t), \theta_i))$ It will satisfy the equation $\dot{e}_i = -\tau e_i$, yielding the target equality: $e_i(t) = e^{-(t-t_0)} e_i(t_0) = 0 \ \forall \ t \geq t_0$.

by equations 3.5 to 3.7 with appropriately chosen parameters and terms $\tau f_i(\xi(t), \theta_i)$ instead of $f_i(\xi(t), \theta_i)$. Hence, even in the absence of actual dynamics, we can still use representations 3.5 to 3.7 for a broad class of signals 3.1, in which the time derivative of $\xi(t)$ is uniformly bounded in t.

3.2 The Class of RNNs. The following set of differential equations defines a recurrent neural network,

$$\dot{x}_j = \sum_{m=1}^{N} c_{j,m}\sigma(\mathbf{w}_{j,m}^T \mathbf{u}(s(t), \xi(t), \mathbf{x}) + b_{j,m}),$$
$$j \in \{1, \ldots, N_x\}, \tag{3.8}$$
$$\mathbf{u}(s(t), \xi(t), \mathbf{x}(t)) = s(t) \oplus \xi(t) \oplus \mathbf{x},$$
$$\mathbf{x} = \mathrm{col}(x_1, \ldots, x_{N_x}),\ \ \mathbf{x}(t_0) = \mathbf{x}_0,$$

where functions $\sigma : \mathbb{R} \to \mathbb{R}$, $\sigma \in \mathcal{C}^0$ are sigmoid.[4] Vectors $\mathbf{c}_j = \mathrm{col}(c_{j,1}, \ldots, c_{j,N})$, $\mathbf{b}_j = \mathrm{col}(b_{j,1}, \ldots, b_{j,N})$ and matrices $\mathbf{W}_j = (\mathbf{w}_{j,1}, \ldots, \mathbf{w}_{j,N})$ are parameters of the RNN. Functions $\xi(t), s(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $s(t), \xi(t) \in \mathcal{C}^0$ are inputs; $\mathbf{x}$ is the state vector, and $\mathbf{x}_0$ is a vector of initial conditions.

Figure 1 shows the general structure of the network; it also illustrates how the network receives information about the original signals $f_i(\xi(t), \theta_i)$. According to notation 3.8, the network maps two functions of time $s(t)$, $\xi(t)$ into the functions $x_1(t, \mathbf{x}_0)$, ..., $x_{N_x}(t, \mathbf{x}_0)$, which are the solutions of system 3.8. In what follows, we consider variables $s(t)$, $\xi(t)$ as inputs to the network. The function of the first input, $s(t)$, is to communicate information about the signal to be classified. The function of the second input, $\xi(t)$, is to provide the network with enough information to ensure that classification is successful. Our choice of inputs is motivated by known results from systems and control theory; if a system adapts to a class of external signals, it must contain a subsystem, or internal model, that is capable of generating all input signals (Conant & Ashby, 1970; Sontag, 2003). In our case, this corresponds to the case where the network is to model the functions $f_i(\xi(t), \theta_i)$. Hence, in general, knowledge of $\xi(t)$ is necessary for successful classification.[5]

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[4]A function $\sigma : \mathbb{R} \to \mathbb{R}$ is called sigmoid if $\lim_{z\to\infty} \sigma(z) = 1$ and $\lim_{z\to-\infty} \sigma(z) = 0$. Choosing the function σ in notation 3.8 in the class of sigmoid functions is not critical for our analysis. In our proof, we will require only that the sums $\sum_{i=1}^{N} c_i\sigma(\mathbf{w}_i^T \mathbf{u} + b_i)$, $c_i,\ b_i \in \mathbb{R}$, $\mathbf{w}_i \in \mathbb{R}^{N_x+2}$ are dense in $C^0([0, 1]^{N_x+2})$. Hence, any continuous function σ satisfying this requirement can be used in notation 3.8. Detailed discussion and specification of such functions can be found in Chen and Chen (1995).

[5]In principle we could have refrained from using input $\xi(t)$ in our system under the assumption that such a signal can be generated by a fraction of the RNNs' internal states. This, however, would unnecessarily complicate our analysis and increase the number of

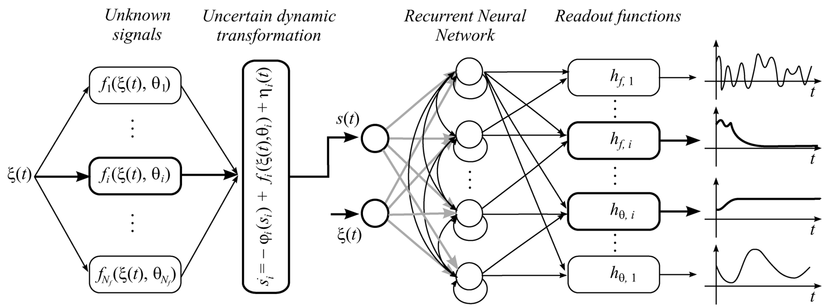

Figure 1: Structure of the network and routing of the signals. The network receives signals $s(t)$, $\xi(t)$ on its inputs and maps these functions into the trajectories of its state. Classification decisions are communicated through the state readout functions. The latter can in principle be realized by a feedforward component. Here we restrict our consideration to mere existence of these functions (see the statement of the problem in section 3.3).

3.3 Assumptions and Statement of the Problem. While the variable $\xi(t)$ is known a priori, variable $s(t)$ is allowed to vary within the set of functions $s_i(t, s_{i,0}, \theta_i, \eta_i(t))$, which are the solutions of equation 3.5. In particular, we assume that the following condition is satisfied:

Assumption 1 (Existence). *There exist $i \in N_f$, $\theta_i \in \Omega_\theta$, $s_{i,0} \in \Omega_s$, and $\eta_i(t)$ specified by equation 3.6 such that*

$$s(t) = s_i(t, s_{i,0}, \theta_i, \eta_i(t)) \; \forall \, t \geq 0. \tag{3.9}$$

We aim to determine if there is a network of type 3.8 that is able to recover uncertain parameters i and θ_i from the input $s(t)$,[6] $t \geq t_0 \in \mathbb{R}_{\geq 0}$ within a finite interval of time for all $t_0 \in \mathbb{R}_{\geq 0}$. Informally, this means that there exist two sets of functions of network state $\mathbf{x}$ and input $s(t)$:

$$\begin{aligned}&\{h_{f,j}(\mathbf{x}(t), s(t))\}, \;\; \{h_{\theta,j}(\mathbf{x}(t), s(t))\}\\ &h_{f,j} : \mathbb{R}^{N_x} \times \mathbb{R} \rightarrow \mathbb{R}, \; h_{\theta,j} : \mathbb{R}^{N_x} \times \mathbb{R} \rightarrow \mathbb{R}, \;\; j \in \{1, \ldots, N_f\},\end{aligned} \tag{3.10}$$

technical assumptions. Therefore, we decided to use inputs $s(t)$, $\xi(t)$ instead of just $s(t)$. Nevertheless, we should keep in mind such a possibility when deemed appropriate.

[6] Because filters 3.5 are convergent (Pavlov et al., 2006), the effect of uncertainty in parameter $s_{i,0}$ vanishes with time exponentially. Hence, the only effective uncertainties are i and θ_i.

such that the values of i and θ_i can be inferred from $\{h_{f,j}(\mathbf{x}(t), s(t))\}$, $\{h_{\theta,j}(\mathbf{x}(t), s(t))\}$, respectively, within a given finite interval of time. Formally we can state this as follows:

Problem 1. *Consider class $\mathcal{F}$ of signals 3.1, where the function $\xi(t)$ is known, and the values of parameters θ_i are unknown a priori. Determine a recurrent neural network 3.8 such that the following properties hold:*

1. *There is a set of initial conditions Ω_x such that $\mathbf{x}(t, \mathbf{x}_0)$ is bounded for all $\mathbf{x}_0 \in \Omega_x$ and $t \geq t_0 \in \mathbb{R}_{\geq 0}$; the volume of Ω_x is nonzero.*
2. *There exists a set of output functions 3.10 such that for all $\theta_i \in \Omega_\theta$, $s_{i,0} \in \Omega_s$, $t_0 \in \mathbb{R}_{\geq 0}$, $\mathbf{x}_0 \in \Omega_x$, and functions $\eta_i(t)$ given by equations 3.6, condition 3.9 implies existence of a constant $\mathcal{T} \in \mathbb{R}_{>0}$, time instant $t' \in (t_0, t_0 + \mathcal{T})$, (arbitrarily large) $T^* \in \mathbb{R}_{>0}$, and (arbitrarily small) $\varepsilon \in \mathbb{R}_{>0}$ and $\mathcal{D} \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} &\|h_{f,i}(\mathbf{x}(t), s(t))\|_{\infty,[t',t'+T^*]} < \varepsilon + \mathcal{D}(\Delta_\eta), \\ &\inf_{\theta_i' \in E(\theta_i)} \|h_{\theta,i}(\mathbf{x}(t), s(t)) - \theta_i'\|_{\infty,[t',t'+T^*]} < \varepsilon + \mathcal{D}(\Delta_\eta). \end{aligned} \tag{3.11}$$

The functions $h_{f,i}(\cdot,\cdot)$, $h_{\theta,i}(\cdot,\cdot)$ in the problem are defined by equation 3.10. The existence of these functions implies a simple readout mechanism for recovering variables i and θ_i. When input $s(t)$ is generated by a signal from the ith class, then $h_{f,i}(\mathbf{x}(t), s(t))$ must be in the neighborhood of zero for sufficiently long time T^*, and the function $h_{f,i}(\mathbf{x}(t), s(t))$ should be in the vicinity of some constant over the same time interval (see Figure 1 for illustration). Furthermore, as follows from equation 3.11, the value of θ_i is estimated by $h_{\theta,i}(\mathbf{x}(t), s(t))$. Hence, information about the class and parameters of an input signal can be inferred from the values of $h_{f,i}(\mathbf{x}(t), s(t))$, $h_{\theta,i}(\mathbf{x}(t), s(t))$. In case inequality 3.11 holds for multiple indices i, additional validation might be necessary. This, however, is beyond the scope of our study. In our current work, we wish to determine if the desired behavior specified by inequality 3.11 can in principle be realized in the RNNs.

In general, the problem has no solutions for all possible functions $\xi(t) \in \mathcal{C}^1$, $f_i(\cdot,\cdot) \in \mathcal{C}^0$ and every $\theta_i \in \Omega_\theta$. Consider, for instance, the case where $f_i(\xi(t), \theta_i) = \sin(\xi(t)\theta_i)$ and

$$\xi(t) = \begin{cases} \sin^2(\ln(t - t_0 + 1)), & \sin(\ln(t - t_0 + 1)) \geq 0 \\ 0, & \sin(\ln(t - t_0 + 1)) < 0 \end{cases} \quad \forall\, t \geq t_0.$$

Time intervals in which $\xi(t) = 0$ are growing unboundedly in length with time. Hence, for any fixed $\mathcal{T}$, T^*, there will always exist a time instant t_0' such that for all $t \geq t_0'$, the lengths of intervals when $\xi(t) = 0$ exceed $\mathcal{T} + T^*$. For all such intervals $\mathcal{T}_j$, $j = 1, \ldots, \infty$ and every $\theta_i \in \Omega_\theta$, it holds that

$f_i(\xi(t), \theta_i) = 0$. This implies that solutions $s_i(t, s_{i,0}, \theta_i, \eta_i(t))$ do not depend on θ_i for all $t \in \mathcal{T}_j$.[7] Hence, recovery of the actual values of θ_i from signal $s(t)$ cannot be achieved within a fixed time interval $[t_0, t_0 + \mathcal{T} + T^*]$ for all $t_0 \geq t'_0$. In order to enable a solution of the classification and recognition problem above, we must introduce an additional constraint on the functions $f_i(\xi(t), \theta_i)$. This constraint should ensure that variation in parameter θ_i can be detected from the values of $f_i(\xi(t), \theta_i)$ within a finite time interval. We therefore require that the following property holds:

Assumption 2 (Nondegeneracy). *For the set of functions $f_i(\xi(t), \theta_i)$ specified by equation 3.1 and all $t \geq t_0$, θ_i, θ'_i, there exist a constant $T \in \mathbb{R}_{>0}$ and a strictly increasing function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\rho \in \mathcal{K}_\infty$ such that the following condition holds:*

$$\forall\, t \geq t_0\ \exists\, t' \in [t, t+T] : \quad |f_i(\xi(t'), \theta_i) - f_i(\xi(t'), \theta'_i)| \geq \rho(\|\theta_i\|_{E_i(\theta'_i)}). \tag{3.12}$$

In case the equivalence classes $E_i(\theta'_i)$ consist of single elements, for example, when there is a unique value of $\theta'_i = \theta_i$ satisfying equation 3.2, condition 3.12 will have a more transparent form:

$$\forall\, t \geq t_0\ \exists\, t' \in [t, t+T] : \quad |f_i(\xi(t'), \theta_i) - f_i(\xi(t'), \theta'_i)| \geq \rho(|\theta_i - \theta'_i|). \tag{3.13}$$

These conditions simply state that within a fixed time interval, the values of $\|\theta_i\|_{E_i(\theta'_i)}$ or $|\theta_i - \theta'_i|$ can be inferred from the differences $f_i(\xi(t), \theta_i) - f_i(\xi(t), \theta'_i)$ for all $t \in \mathbb{R}_{\geq 0}$.

In the next section, we show that a solution to the problem can be obtained for the class $\mathcal{F}$ of functions $f_i(\xi(t), \theta_i)$ that are Lipschitz in θ_i. We present these results in the form of sufficient conditions formulated in theorem 1. In addition to showing the existence of an RNN and corresponding functions 3.10 satisfying the requirements of the problem, we demonstrate that functions 3.10 can be made continuous and differentiable. Hence, in principle, they can be implemented as an extra feedforward component in the existing RNN structure, 3.8.

4 Main Results

As suggested in our previous work (Prokhorov et al., 2002), as well as in Younger et al. (1999), the reason that RNNs with fixed parameters (i.e.,

[7] This is, of course, not true when initial conditions $s_{i,0}$ are dependent on θ_i and such dependence is known a priori. In our work, due to the presence of noise $\eta_i(t)$ and potential uncontrolled changes of θ_i for $t < t_0$, we do not consider this special case.

weights) demonstrate adaptive behavior can be found in their dynamics; supposedly it is already sufficiently rich to have an adequate adaptation mechanism embedded in it. Finding a system that satisfies requirements 1 and 2 in the problem and is, at the same time, realizable by an RNN therefore automatically constitutes an existence proof. This intuition, we will show, is correct. The result is provided in theorem 1:

Theorem 1 (Existence). *Let functions $\xi(t)$, $f_i(\xi(t), \theta_i)$ be given and defined as in equation 3.1 and assumptions 1 and 2 hold. Furthermore, suppose that $f_i(\xi(t), \theta_i)$ are (locally) Lipschitz:*[8]

$$\exists\, D_\theta \in \mathbb{R}_{>0} : \ |f_i(\xi(t), \theta_i) - f_i(\xi(t), \theta_i')| \le D_\theta \, \|\theta_i\|_{E_i(\theta_i')} \qquad \forall\, t > 0,\ \theta_i, \theta_i' \tag{4.1}$$

$$\exists\, D_\xi \in \mathbb{R}_{>0} : \ |f_i(\xi, \theta_i) - f_i(\xi', \theta_i)| \le D_\xi |\xi - \xi'| \quad \forall\, \theta_i, \xi, \xi' \tag{4.2}$$

and the time derivative of $\xi(t)$ is bounded:

$$\left| \frac{d}{dt}\xi(t) \right| \le \partial\xi_\infty \ \forall\, t \ge 0, \tag{4.3}$$

then for any $T^* \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$ there is a recurrent neural network 3.8 satisfying the requirements of problem 1, provided that the upper bound Δ_η for the $L_\infty[0, \infty]$-norms of the disturbance terms, $\eta_i(t)$, is sufficiently small.

Proof. We prove the theorem in four steps. First, we present a dynamical system, which will be referred to as the *convergence prototype*. We have chosen this system to belong to the following class of differential algebraic equations:

$$\dot{\hat{s}}_i = -\varphi_i(\hat{s}_i) + f_i(\xi(t), \hat{\theta}_i) \tag{4.4}$$

$$\hat{\theta}_i = a + \frac{b-a}{2}(x_i + 1) \tag{4.5}$$

$$\begin{aligned} \dot{x}_i &= \gamma \|\hat{s}_i - s\|_\varepsilon \big(x_i - y_i - x_i(x_i^2 + y_i^2)\big) \\ \dot{y}_i &= \gamma \|\hat{s}_i - s\|_\varepsilon \big(x_i + y_i - y_i(x_i^2 + y_i^2)\big), \end{aligned} \tag{4.6}$$

where

$$\gamma \in \mathbb{R}_{>0},\ a, b \in \mathbb{R},\ a < \theta_{\min},\ b > \theta_{\max},\ \theta_0 \in [a, b], \qquad i = 1, \ldots, N_f,\ \varepsilon \in \mathbb{R}_{>0}. \tag{4.7}$$

[8]Property 4.1 can be understood as a generalized Lipschitz condition. When equivalence sets $E_i(\theta_i')$ consist of single elements, the property transforms into $|f_i(\xi(t), \theta_i) - f_i(\xi(t), \theta_i')| \le D_\theta |\theta_i - \theta_i'|$.

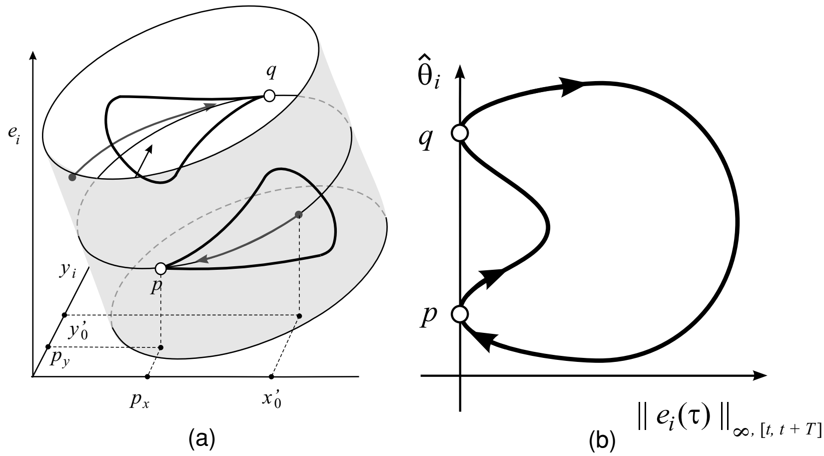

Figure 2: Diagrams illustrating invariant and limit sets of system 4.4 to 4.6 (a), $e_i = s_i - \hat{s}_i$) and schematic representation of the system dynamics in terms of $\hat{\theta}_i(x_i(t))$, $\|e_i(\tau)\|_{\infty,[t,t+T]}$ (b). (a) Invariant and limit sets of system 4.4 to 4.6, shown as circles p, q. Gray curves show trajectories that converge to these sets asymptotically. Domains of attraction of these sets, restricted to the cylinder $x_i^2 + y_i^2 = 1$, are shown by thick lines on the surface of the cylinder. Notice that these domains are not neighborhoods of p, q. (b) Schematic depiction of the dynamics of system 4.4 to 4.6: traveling along an attracting closed orbit until the domain of attraction of either p or q is reached.

Notice that is is always possible to choose parameter values of system 4.4 to 4.6 in accordance with 4.7. Indeed, the fact that $\theta_0 \in \mathbb{R}$ implies the existence of an interval $[a, b] \subset \mathbb{R}$ such that $\theta_0 \in [a, b]$ and $[\theta_{\min}, \theta_{\max}] \subset [a, b]$.

System 4.4 to 4.6 has a locally Lipschitz right-hand side, and its solutions are bounded for all initial conditions $\hat{s}_i(t_0)$, $x_i(t_0)$, $y_i(t_0) \in \mathbb{R}$. Equation 4.4 models the dynamics of input signal $s(t)$; this requirement is shown to be generally necessary for successful classification and adaptation (Conant & Ashby, 1970; Sontag, 2003). The intuition behind equations 4.5 and 4.6 is as follows. For every $\theta_i \in \Omega_\theta$ and in the absence of $\eta_i(t)$, there will always exist a point $x_i(t_0) = p_x$, $y_i(t_0) = p_y$, $p_x^2 + p_y^2 = 1$ such that $\hat{\theta}_i(x_i(t, p_x, p_y)) \in E_i(\theta_i)$ for all $t \geq t_0$ provided that $s(t) = s_i(t, s_{i,0}, \theta_i, 0)$ (see Figure 2). When signals $\eta_i(t)$ are present, we will show that $\hat{\theta}_i(x_i(t, p_x, p_y))$ will remain in the neighborhood of $E_i(\theta_i)$, of which the size is determined by Δ_η. When $|s(t) - \hat{s}_i(t)| > \varepsilon$, solutions of equations 4.5 and 4.6, starting from initial conditions $x_i(t_0) = x_0'$, $y_i(t_0) = y_0'$, $x_0'^2 + y_0'^2 = 1$ evolve along a closed orbit toward p_x, p_y. We show that there exist $\gamma > 0$, $\varepsilon > 0$ and their respective domains, and a point $\hat{s}_i(t_0) = s_0'$, $x_i(t_0) = x_0'$, $y_i(t_0) = y_0'$, such that the trajectories passing

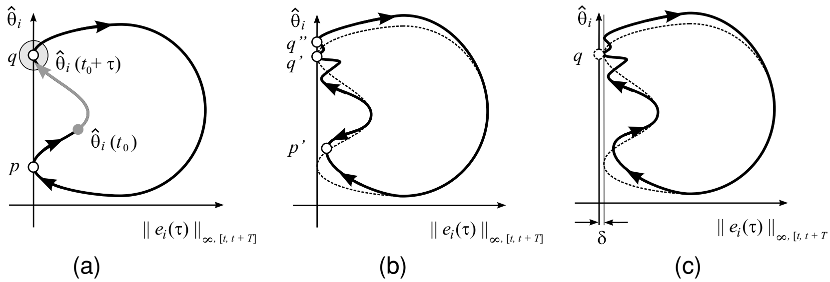

Figure 3: Diagrams illustrating the main steps of the proof. (a) Corresponds to the case, without perturbations and approximation errors, in which estimate $\hat{\theta}_i(t)$, shown as a gray curve, asymptotically approaches points from a neighborhood of the equivalence class $E_i(\theta_i)$ for a given set of initial conditions and all θ_i. Existence of such systems (e.g., 4.4 to 4.6) constitutes the first step of our proof. In the second step, we show that the time required to reach this neighborhood for a given initial condition and all $\theta_i \in \Omega_\theta$ is bounded from above by some positive number τ. (b) Corresponds to the case where the convergence prototype is approximated by a recurrent neural network. Because system 4.4 to 4.6 is not structurally stable, approximation errors might lead to the emergence of new attracting invariant sets (points p', q', and q'' in the figure) that do not belong to the neighborhood of $E_i(\theta_i)$. The diagram of system's behavior in this case is shown by a thick, solid line. (c) Illustrates the dynamics of the perturbed yet structurally stable system 4.9, 4.10. In this case, convergence to invariant sets p, q is replaced with arbitrarily slow relaxation in small neighborhoods of these points (ghost attractors). Even in the presence of disturbances due to approximation errors, the system's state still visits these sets (shown by the dashed circle) and stays there as long as needed. The third step of the proof demonstrates this.

through this point converge to the following target set:

$$\|\hat{s}_i - s_i\|_\varepsilon = 0, \quad \left\|\hat{\theta}_i\right\|_{E_i(\theta_i)} \leq \varepsilon_\theta(\varepsilon). \tag{4.8}$$

Second, we prove that there is a point $x_i(t_0) = x'_0$, $y_i(t_0) = y'_0$ such that convergence is locally uniform with respect to the values of uncertain parameters θ_i and $s_{i,0}$. In other words, for all $t_0 \geq 0$, $s_{i,0} \in \Omega_s$, and $\theta_i \in \Omega_\theta$, there exists $\tau > 0$ such that solutions of system 4.4 to 4.6 with initial conditions $x_i(t_0) = x'_0$, $y_i(t_0) = y'_0$ will be in an arbitrarily small neighborhood of equation 4.8 for all $t \geq t_0 + \tau$ (see also Figure 3).

System 4.4 to 4.6, however, is not structurally stable. That is, small perturbations of its right-hand side might change the asymptotic properties of the system drastically. Hence, due to the inevitable approximation errors, the chances that an RNN realization of this system would solve problem 1 are slim. To continue our argument, we need to modify system 4.4 to 4.6 such that the resulting system becomes structurally stable.

For this reason, we, third, consider the perturbed version of system 4.4 to 4.6:

$$\begin{aligned} \dot{\hat{s}}_i &= -\varphi_i(\hat{s}_i) + f_i\left(\xi(t), \hat{\theta}_i\right) \\ \hat{\theta}_i &= a + \tfrac{b-a}{2}(x_i + 1) \end{aligned} \tag{4.9}$$

$$\begin{aligned} \dot{x}_i &= \gamma(\|\hat{s}_i - s\|_\varepsilon + \delta)\left(x_i - y_i - x_i\left(x_i^2 + y_i^2\right)\right) \\ \dot{y}_i &= \gamma(\|\hat{s}_i - s\|_\varepsilon + \delta)\left(x_i + y_i - y_i\left(x_i^2 + y_i^2\right)\right), \ \ \delta \in \mathbb{R}_{>0}. \end{aligned} \tag{4.10}$$

Our aim is to achieve structural stability of an otherwise structurally unstable system. We show that trajectories of system 4.9, 4.10 periodically visit a small vicinity of equation 4.8 and stay there for an arbitrary long time, depending on the value of δ. Fourth, given that system 4.9, 4.10 is structurally stable, we apply the results from Cybenko (1989) to demonstrate that solutions of system 4.9, 4.10 can be approximated in forward time over the semi-infinite interval $[0, \infty]$ by the state of a recurrent neural network specified by equations 3.8.

Figure 3 provides diagrams linking the main steps of the proof:

1. Convergence prototype. According to assumption 1, there exist $i \in \{1, \ldots, N_f\}$, $s_{i,0}$, θ_i such that $s(t) = s_i(t, s_{i,0}, \theta_i, \eta_i(t))$ for all $t \geq 0$. Consider the ith subsystem of 4.4 to 4.6, and analyze the dynamics of the following difference: $s_i(t) - \hat{s}_i(t)$:

$$\begin{aligned} \frac{d}{dt}(s_i(t) - \hat{s}_i(t)) = &-(\varphi_i(s_i) - \varphi_i(\hat{s}_i)) + f_i(\xi(t), \theta_i) \\ &- f_i(\xi(t), \hat{\theta}_i(x_i(t))) + \eta_i(t). \end{aligned}$$

According to our assumptions, functions $\varphi_i(\cdot)$ are differentiable. Hence, invoking Hadamard's lemma,

$$\varphi_i(s_i) - \varphi_i(\hat{s}_i) = \left(\int_0^1 \frac{\partial \varphi(s_i r + (1-r)\hat{s}_i)}{\partial(s_i r + (1-r)\hat{s}_i)} dr\right)(s_i - \hat{s}_i), \tag{4.11}$$

and denoting

$$\begin{aligned} e_i(t) &= s(t) - \hat{s}_i(t) = s_i(t) - \hat{s}_i(t), \\ \alpha_i(t) &= \int_0^1 \frac{\partial \varphi(s_i(t) r + (1-r)\hat{s}_i(t))}{\partial(s_i(t) r + (1-r)\hat{s}_i(t))} dr, \\ \Delta f_i(t) &= f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i(x_i(t))), \end{aligned} \tag{4.12}$$

we can obtain the following equivalent representation of equation 4.11:

$$\dot{e}_i = -\alpha(t) e_i + \Delta f_i(t) + \eta_i(t). \tag{4.13}$$

Solutions of equation 4.13 satisfy expression

$$e_i(t) = e^{-\int_0^t \alpha(\tau)d\tau} e_i(0) + e^{-\int_0^t \alpha(\tau)d\tau} \int_0^t e^{\int_0^\tau \alpha(\tau_1)d\tau_1} (\Delta f_i(\tau) + \eta_i(\tau))d\tau.$$

According to the mean value theorem, the following equivalence holds: $\alpha(t) = \partial\varphi(s_i(t)r' + (1-r')\hat{s}_i(t))/\partial(s_i(t)r' + (1-r')\hat{s}_i(t))$ for some $r' \in [0, 1]$. Hence taking condition 3.7 into account, we can obtain that $\varphi_{\min} \le \alpha(t) \le \varphi_{\max}$. Therefore, using inequality

$$e^{-\int_0^t \alpha(\tau)d\tau} \le e^{-\varphi_{\min} t},$$

we can derive the following estimate:

$$\begin{aligned}
|e_i(t)| \le\, & e^{-\int_0^t \alpha_i(\tau)d\tau} |e_i(0)| + \Bigg| e^{-\int_0^t \alpha(\tau)d\tau} \\
& \times \int_0^t e^{\int_0^\tau \alpha(\tau_1)d\tau_1} (\Delta f_i(\tau) + \eta_i(\tau))d\tau \Bigg| \\
\le\, & e^{-\varphi_{\min} t} |e_i(0)| + e^{-\int_0^t \alpha(\tau)d\tau} \int_0^t e^{\int_0^\tau \alpha(\tau_1)d\tau_1} \left|\Delta f_i(\tau) + \eta_i(\tau)\right| d\tau \\
=\, & e^{-\varphi_{\min} t} |e_i(0)| + \int_0^t e^{-\int_\tau^t \alpha(\tau_1)d\tau_1} \left|\Delta f_i(\tau) + \eta_i(\tau)\right| d\tau \qquad (4.14) \\
\le\, & e^{-\varphi_{\min} t} |e_i(0)| + \int_0^t e^{-\varphi_{\min}(t-\tau)} \left|\Delta f_i(\tau) + \eta_i(\tau)\right| d\tau \\
\le\, & e^{-\varphi_{\min} t} |e_i(0)| + \frac{1}{\varphi_{\min}} \left(1 - e^{-\varphi_{\min} t}\right) \\
& \times (\|\Delta f_i(\tau)\|_{\infty,[0,t]} + \|\eta_i(\tau)\|_{\infty,[0,\infty]}).
\end{aligned}$$

Given that $\|\eta_i(\tau)\|_{\infty,[0,\infty]} \le \Delta_\eta$ for all $t \ge 0$, inequality 4.14 implies that

$$\begin{aligned}
|e_i(t)| &\le e^{-\varphi_{\min} t} |e_i(0)| + \tfrac{1}{\varphi_{\min}} (1 - e^{-\varphi_{\min} t}) (\|\Delta f_i(\tau)\|_{\infty,[0,t]} + \Delta_\eta) \\
&= e^{-\varphi_{\min} t} \left(|e_i(0)| - \tfrac{\Delta_\eta}{\varphi_{\min}}\right) + \tfrac{1}{\varphi_{\min}} (\|\Delta f_i(\tau)\|_{\infty,[0,t]} + \Delta_\eta).
\end{aligned} \qquad (4.15)$$

Regrouping terms in equation 4.15 yields

$$\left(|e_i(t)| - \frac{\Delta_\eta}{\varphi_{\min}}\right) \le e^{-\varphi_{\min} t} \left(|e_i(0)| - \frac{\Delta_\eta}{\varphi_{\min}}\right) + \frac{1}{\varphi_{\min}} \|\Delta f_i(\tau)\|_{\infty,[0,t]}.$$

Let us denote $\varepsilon = \Delta_\eta/\varphi_{\min}$ and consider the values of $\|e_i(t)\|_\varepsilon$. When $|e_i(t)| - \Delta_\eta/\varphi_{\min} > 0$, we have

$$\|e_i(t)\|_\varepsilon = \left(|e_i(t)| - \frac{\Delta_\eta}{\varphi_{\min}}\right) \leq e^{-\varphi_{\min} t}\|e_i(0)\|_\varepsilon + \frac{1}{\varphi_{\min}}\|\Delta f_i(\tau)\|_{\infty,[0,t]}.$$

When $|e_i(t)| - \Delta_\eta/\varphi_{\min} \leq 0$, then

$$\|e_i(t)\|_\varepsilon = 0 \leq e^{-\varphi_{\min} t}\|e_i(0)\|_\varepsilon + \frac{1}{\varphi_{\min}}\|\Delta f_i(\tau)\|_{\infty,[0,t]}.$$

Hence, we can conclude that the following estimate holds along the trajectories of equation 4.4:

$$\|e_i(t)\|_\varepsilon \leq e^{-\varphi_{\min} t}\|e_i(0)\|_\varepsilon + \frac{1}{\varphi_{\min}}\|\Delta f_i(\tau)\|_{\infty,[0,t]}, \quad \varepsilon = \frac{\Delta_\eta}{\varphi_{\min}}. \tag{4.16}$$

Taking equations 4.1 and 4.16 into account, plus the fact that $\left\|\hat{\theta}_i\right\|_{E_i(\theta_i)} = \inf_{\bar{\theta}_i \in E_i(\theta_i)} |\hat{\theta}_i - \bar{\theta}_i|$, we can conclude that the following inequality holds:

$$\begin{aligned}\|e_i(t)\|_\varepsilon \leq{}& e^{-\varphi_{\min} t}\|e_i(0)\|_\varepsilon \\ &+ \frac{D_\theta}{\varphi_{\min}}\|\bar{\theta}_i - \hat{\theta}_i(\tau)\|_{\infty,[0,t]},\ \bar{\theta}_i \in E_i(\theta_i) \cap [a, b].\end{aligned} \tag{4.17}$$

Let us now consider equations 4.5 and 4.6. We pick up a point x', y' that satisfies the following condition:

$$x'^2 + y'^2 = 1. \tag{4.18}$$

Solutions of equation 4.6 passing through this point can be defined as follows:

$$\begin{aligned}x_i(t, x', y') &= \cos\left(\int_0^t \gamma\|\hat{s}_i(\tau) - s(\tau)\|_\varepsilon d\tau + \nu_x\right),\ x' = \cos(\nu_x),\ \nu_x \in [0, 2\pi] \\ y_i(t, x', y') &= \sin\left(\int_0^t \gamma\|\hat{s}_i(\tau) - s(\tau)\|_\varepsilon d\tau + \nu_y\right),\ y' = \sin(\nu_y),\ \nu_y \in [0, 2\pi].\end{aligned} \tag{4.19}$$

This observation can easily be verified when writing equation 4.6 in polar coordinates: $x_i = r\cos(\nu)$, $y_i = r\sin(\nu)$ (Guckenheimer and Holmes, 2002):

$$\begin{aligned}\dot{r} &= \gamma\|\hat{s}_i - s\|_\varepsilon \cdot r(1 - r) \\ \dot{\nu} &= \gamma\|\hat{s}_i - s\|_\varepsilon \cdot 1.\end{aligned} \tag{4.20}$$

Given that $\bar{\theta}_i$ belongs to the interval $[a, b]$, there is a number $\bar{h}(\bar{\theta}_i) \in [0, \pi]$ such that for all $k \in \mathbb{Z}$, the following equivalence holds:

$$\bar{\theta}_i = a + \frac{b-a}{2}\left(\cos(\bar{h}(\bar{\theta}_i) + 2\pi k) + 1\right). \tag{4.21}$$

Hence, according to equation, 4.5 and 4.19, the norm $\|\bar{\theta}_i - \hat{\theta}_i(\tau)\|_{\infty,[0,t]}$ can be estimated from the above as follows:

$$\begin{aligned}\|\bar{\theta}_i - \hat{\theta}_i(\tau)\|_{\infty,[0,t]} \leq \frac{b-a}{2}\|\bar{h}(\bar{\theta}_i) - \nu_x + 2\pi k \\ - \int_0^t \gamma\|\hat{s}_i(\tau) - s(\tau)\|_\varepsilon d\tau\|_{\infty,[0,t]}.\end{aligned} \tag{4.22}$$

Denoting

$$c = \frac{D_\theta}{\varphi_{\min}}\frac{b-a}{2}; \quad h(t, \bar{\theta}_i, k) = \bar{h}(\bar{\theta}_i) - \nu_x + 2\pi k - \int_0^t \gamma\|\hat{s}_i(\tau) - s(\tau)\|_\varepsilon d\tau$$

and taking into account equations 4.17 and 4.22, we can conclude that the following holds along the solutions of system 4.4 to 4.6:

$$\begin{aligned}&\|e_i(t)\|_\varepsilon \leq e^{-\varphi_{\min} t}\|e_i(0)\|_\varepsilon + c\|h(\tau, \bar{\theta}_i, k)\|_{\infty,[0,t]};\\ &h(0, \bar{\theta}_i, k) - h(t, \bar{\theta}_i, k) = \int_0^t \gamma\|e_i(\tau)\|_\varepsilon d\tau.\end{aligned} \tag{4.23}$$

According to Tyukin et al. (2008, theorem 1 and corollaries 2 and 3) there exist $\gamma^* \in \mathbb{R}_{>0}$ and h^* such that for a given bounded $e_i(0)$, all $\gamma \in \mathbb{R}_{>0}, \gamma < \gamma^*$ and $h(0, \bar{\theta}_i, k) \geq h^*$ the norm $\|e_i(\tau)\|_{\infty,[0,\infty]}$ is bounded and

$$\lim_{t\to\infty} h(t, \bar{\theta}_i, k) \in [0, h(0, \bar{\theta}_i, k)]. \tag{4.24}$$

The value of γ^* can be determined, according to corollary 3 in Tyukin et al. (2008), from the following inequality:

$$\begin{aligned}&0 < \gamma^* < \frac{\varphi_{\min}}{c}\left(\ln\left(\frac{\kappa}{d}\right)\frac{\kappa}{\kappa-1}\left(2 + \frac{\kappa}{1-d}\right)\right)^{-1},\\ &\qquad \kappa \in \mathbb{R}_{>1},\ d \in (0,1) \subset \mathbb{R}.\end{aligned} \tag{4.25}$$

The value of h^* can be estimated from

$$\|e_i(t_0)\|_\varepsilon \leq \left(\frac{\varphi_{\min}}{\gamma^*}\left(\ln\frac{\kappa}{d}\right)^{-1}\frac{\kappa-1}{\kappa} - c\left(2 + \frac{\kappa}{1-d}\right)\right)h^*. \tag{4.26}$$

Given that $\|e_i(t_0)\|_\varepsilon$ in equation 4.26 is bounded from above for all $t_0 \geq 0$, $\|e_i(t_0)\|_\varepsilon \leq s_{\max} - s_{\min} + D_\theta/\varphi_{\min}(b-a)$, condition

$$h^* \geq \left((s_{\max} - s_{\min}) + \frac{D_\theta(b-a)}{\varphi_{\min}}\right) \times \left(\frac{\varphi_{\min}}{\gamma^*}\left(\ln\frac{\kappa}{d}\right)^{-1}\frac{\kappa-1}{\kappa} - c\left(2 + \frac{\kappa}{1-d}\right)\right)^{-1}, \tag{4.27}$$

together with equation 4.25, imply that for all $\hat{s}_i(t_0) \in \Omega_s$ and $h(0, \bar{\theta}_i, k) \geq h^*$, the norm $\|e_i(\tau)\|_{\infty,[0,\infty]}$ is bounded and property 4.24 holds.

Notice that in the definition of $h(0, \bar{\theta}_i, k)$,

$$h(0, \bar{\theta}_i, k) = \bar{h}(\bar{\theta}_i) - \nu_x + 2\pi k, \tag{4.28}$$

the value of k can be chosen arbitrarily large. Moreover, $\bar{h}(\bar{\theta}_i) \in [0, \pi]$ for all $\bar{\theta}_i \in [a, b]$. This implies that there exists a finite k' such that condition $h(0, \bar{\theta}_i, k') \geq h^*$ will be satisfied for any fixed h^* (i.e., for all γ^* satisfying equation 4.25) and all $\bar{\theta}_i \in [a, b]$. In addition, the following will hold:

$$\lim_{t\to\infty} h(t, \bar{\theta}_i, k') \in [0, h(0, \bar{\theta}_i, k')] \subset [0, \pi - \nu_x + 2\pi k'] \quad \forall \in \bar{\theta}_i \in [a, b]. \tag{4.29}$$

Taking equation 4.19 into account, we can conclude that solutions $x_i(t, x', y')$ converge to a point in the interval $[-1, 1]$ as $t \to \infty$, and vector $(x_i(t, x', y'), y_i(t, x', y'))$ makes no more than k' full rotations around the origin for all $\theta_i \in [\theta_{\min}, \theta_{\max}]$. Hence for a given initial condition $x_i(0) = x'$, $y_i(0) = y'$, $\hat{s}_{i,0} \in \Omega_s$ and $\theta_i \in [\theta_{\min}, \theta_{\max}]$ the estimate $\hat{\theta}_i(t) = a + (b-a)/2 \cdot (x_i(t, x', y') + 1)$ converges to a point in $[a, b]$ as $t \to \infty$. We denote this point by symbol $\hat{\theta}_i^*$.

Given that $\hat{\theta}_i(t)$ converges to a limit, there exists a time instant t^* such that for all $t \geq t^*$, the following condition holds: $|\hat{\theta}_i(t) - \hat{\theta}_i^*| < \mu_\infty$, where $\mu_\infty \in \mathbb{R}_{>0}$ is an arbitrarily small constant. Therefore, taking condition 4.1 into account, we can conclude that for all $t \geq t^*$, derivative $\dot{e}_i$ satisfies the following equation:

$$\dot{e}_i = -\alpha(t)e_i + f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i^*) + \mu_i(t) + \eta_i(t), \tag{4.30}$$

where $|\mu_i(t)| \leq D_\theta\, \mu_\infty$ is a continuous function.

Now we show that the norm $\|\theta_i\|_{E_i(\hat{\theta}_i^*)}$ can be bounded from above by a $\mathcal{K}_\infty$-function of Δ_η. Consider the term $f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i^*)$. According to equation 3.12, there exists a sequence of monotonically increasing time instances t_j, $j = 1, 2, \ldots$ such that $t_{j+1} - t_j \leq 2T$ and $|f_i(\xi(t_j), \theta_i) - f_i(\xi(t_j), \hat{\theta}_i^*)| \geq \rho(\|\theta_i\|_{E_i(\hat{\theta}_i^*)})$. Furthermore, according to equations 4.2 and 4.3, the time derivative of $f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i^*)$ is bounded:

$$\left| \frac{d}{dt} f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i^*) \right| \leq 2D_\xi \cdot \partial\xi_\infty = D_f.$$

Hence the following estimate holds:

$$\begin{aligned} &\int_t^{t+L} |f_i(\xi(\tau), \theta_i) - f_i(\xi(\tau), \hat{\theta}_i^*)| \geq \frac{\rho\left(\|\theta_i\|_{E_i(\hat{\theta}_i^*)} \right)^2}{2D_f} \\ &L = \max\left\{ 2T, \frac{\rho(b-a)}{D_f} \right\}. \end{aligned} \tag{4.31}$$

In order to proceed further, we need the following lemma:

Lemma 1. *Consider the following differential equation:*

$$\dot{z} = -\varphi(t, z) + u(t) + \eta(t), \; z_0 = z(0) \in [z_{min}, z_{max}] \subset \mathbb{R}. \tag{4.32}$$

Let us suppose that

1. $\varphi(z)z \geq 0, \; \varphi_{min} \leq \partial\varphi(t, z)/\partial z \leq \varphi_{max}$.
2. $u(t) \in L_\infty[0, \infty] \cap \mathcal{C}^1, \; \|u(t)\|_{\infty,[0,\infty]} \leq u_\infty, \|\dot{u}(t)\|_{\infty,[0,\infty]} \leq \partial u_\infty$
3. $\eta(t) \in L_\infty[0, \infty], \|\eta(t)\|_{\infty,[0,\infty]} \leq \Delta$
4. *There exist constants* L, δ, *such that for all* $t \geq 0$.
$$\int_t^{t+L} |u(\tau)| d\tau \geq \delta. \tag{4.33}$$
5. *Finally, assume that the following inequality hold:*
$$\left(\frac{\delta}{L} \right)^2 - \Delta u_\infty > 0. \tag{4.34}$$

Then for any $p \in \mathbb{R}_{>0}$, *there exist constants* $L^* > 0$ *and* $\delta^* \geq ((\delta/L)^2 - \Delta u_\infty)/p$, *such that*

$$\int_t^{t+L^*} |z(\tau)| d\tau \geq \delta^* \geq \frac{1}{p} \left(\frac{\delta^2}{L} - \Delta u_\infty L \right) \forall \, t \geq 0. \tag{4.35}$$

Proof. We prove the lemma along the lines of an argument provided in (Loria, Panteley, Popovic, & Teel, 2003) (property 1). Consider the time derivative of zu:

$$\begin{aligned} \frac{d}{dt}(zu) &= (-\varphi(t, z) + u + \eta)u + z\dot{u} \geq u^2 \\ &\quad - |z| (\varphi_{\max} + \partial u_\infty) - |u|\Delta. \end{aligned} \tag{4.36}$$

According to equation 4.33 for all $t, t_0 \in \mathbb{R}_{\geq 0}, t \geq t_0$, the following inequality holds:

$$z(t)u(t) - z(t_0)u(t_0) \geq \int_{t_0}^{t} u^2(\tau)d\tau - (\varphi_{\max} + \partial u_\infty) \times \int_{t_0}^{t} |z(\tau)|d\tau - \Delta \int_{t_0}^{t} |u(\tau)|d\tau. \tag{4.37}$$

Rearranging terms in equation 4.37 yields

$$(\varphi_{\max} + \partial u_\infty)\int_{t_0}^{t} |z(\tau)|d\tau \geq z(t_0)u(t_0) - z(t)u(t) + \int_{t_0}^{t} u^2(\tau)d\tau - \Delta \int_{t_0}^{t} |u(\tau)|d\tau.$$

Notice that $z(t_0)u(t_0) - z(t)u(t)$ is bounded from below for all $t \geq 0$. We denote this bound by the symbol M. Furthermore, according to the Holder inequality and property 4.33, the following estimate holds for all $t \geq 0$:

$$\frac{\delta^2}{L} \leq \frac{1}{L}\left(\int_{t}^{t+L} |u(\tau)|d\tau\right)^2 \leq \int_{t}^{t+L} u^2(\tau)d\tau.$$

Hence, for all time instances t: $(n+1)L \geq t - t_0 \geq nL$, where n is a positive integer, we have

$$\begin{aligned}(\varphi_{\max} + \partial u_\infty)\int_{t_0}^{t} |z(\tau)|d\tau &\geq M + n\frac{\delta^2}{L} - \Delta \int_{t_0}^{t} |u(\tau)|d\tau \\ &\geq M + n\frac{\delta^2}{L} - (n+1)\Delta u_\infty = (M - \Delta u_\infty L) + n\left(\frac{\delta^2}{L} - \Delta u_\infty L\right).\end{aligned} \tag{4.38}$$

According to the requirements of the lemma, inequality 4.34, the difference $\delta^2/L - \Delta u_\infty L > 0$ is a positive constant. Therefore, there exists $n = n'$ such that the right-hand side of equation 4.38 exceeds some $\delta' = (\delta^2/L - \Delta u_\infty L)/p' \in \mathbb{R}_{>0},\ p' \in \mathbb{R}_{>0}$. Choosing $t' = \min_t\{t - t_0 \geq n'L\}$, we can conclude that

$$(\varphi_{\max} + \partial u_\infty)\int_{t_0}^{t'} |z(\tau)|d\tau \geq \delta'. \tag{4.39}$$

Given that we could choose the value of t_0 arbitrarily in the domain $\mathbb{R}_{\geq 0}$, inequality 4.39 is equivalent to

$$\int_{t}^{t+L^*} |z(\tau)| d\tau \geq \delta^*,$$

where $L^* = t' - t_0$, $\delta^* = \delta'/(\varphi_{\max} + \partial u_\infty) = (\delta^2/L - \Delta u_\infty L)/p$, $p = p'$ $(\varphi_{\max} + \partial u_\infty)$. *The lemma is proven.*

Denoting $f_i(\xi(t), \theta_i) - f_i(\xi(t), \hat{\theta}_i^*) = u(t)$, $\eta_i(t) + \mu_i(t) = \eta(t)$, we can observe that equation 4.30 is of the same class as equation 4.32 in the formulation of lemma 4. Furthermore, the following inequalities hold:

$$\Delta \leq \Delta_\eta + D_\theta\ \mu_\infty;\quad \|u(t)\|_{\infty,[0,\infty]} \leq D_\theta\ \|\theta_i\|_{E_i(\hat{\theta}_i^*)} \leq D_\theta(b-a). \tag{4.40}$$

Notice that the value of μ_∞ in equation 4.40 can be made arbitrarily small because $\hat{\theta}_i(t)$ converges to a limit, and $\hat{\theta}_i^*$ can be chosen from its arbitrarily small vicinity. Let us therefore choose $\hat{\theta}_i^*$ such that $D_\theta \mu_\infty \leq \Delta_\eta$. Hence, in accordance with lemma 4, the following condition,

$$\left(\frac{\rho^2\left(\|\theta_i\|_{E_i(\hat{\theta}_i^*)}\right)}{2D_f L}\right)^2 > 2\Delta_\eta D_\theta(b-a), \tag{4.41}$$

implies the existence of constants L^*, $p \in \mathbb{R}_{>0}$ such that

$$\int_{t}^{t+L^*} |e_i(\tau)| d\tau \geq \frac{1}{p}\left(\left(\frac{\rho^2\left(\|\theta_i\|_{E_i(\hat{\theta}_i^*)}\right)}{2D_f}\right)^2 \frac{1}{L} - \Delta u_\infty L\right)$$
$$= \delta^* > 0\ \ \forall t \geq t^*. \tag{4.42}$$

We now show that the norm $\|\theta_i\|_{E_i(\hat{\theta}_i^*)}$ is bounded from above by a function $\varepsilon_\theta(\Delta_\eta) \in \mathcal{K}_\infty$ for all sufficiently small Δ_η. Let us parameterize Δ_η as follows:

$$\Delta_\eta = \left(\frac{\rho^2(\varepsilon^*)}{2D_f L}\right)^2 \frac{1}{2D_\theta(b-a)},\ \varepsilon^* \in \mathbb{R}_{>0}. \tag{4.43}$$

Parameterization 4.43 is always possible because $\rho(\cdot) \in \mathcal{K}_\infty$. For all $\|\theta_i\|_{E_i(\hat{\theta}_i^*)} > \varepsilon^*$, condition 4.41 is satisfied. Hence, according to lemma 4, there exist constants L^*, p such that inequality 4.42 holds. Given that δ^*, L^*, $\varphi_{\min} \in \mathbb{R}_{>0}$ there will always exist a number $\Delta_\eta^* \in \mathbb{R}_{>0}$ such that $\Delta_\eta^* < (L^*)^{-1}\delta^*\varphi_{\min}/2$. This implies that for all $\Delta_\eta \leq \Delta_\eta^*$, the following inequality holds:

$$\int_{t}^{t+L^*} \|e_i(\tau)\|_{\varepsilon} d\tau \geq \frac{\delta^*}{2}, \ \varepsilon = \frac{\Delta_\eta}{\varphi_{\min}}. \tag{4.44}$$

Let us suppose that the norm $\|\theta_i\|_{E_i(\hat{\theta}_i^*)}$ is greater than ε^*. In this case equations 4.41 and 4.44 hold, and the integral

$$\int_{t^*}^{t} \|e_i(\tau)\|_{\varepsilon} d\tau \tag{4.45}$$

grows unboundedly with t. On the other hand, according to equations 4.23 and 4.24, integral 4.45 is bounded. Hence, we have reached a contradiction. This implies that $\|\theta_i\|_{E_i(\hat{\theta}_i^*)} \leq \varepsilon^*$. Given that $\rho(\cdot) \in \mathcal{K}_\infty$, the inverse $\rho^{-1}(\cdot)$ is well defined and is a $\mathcal{K}_\infty$-function. Therefore, taking equation 4.43 into account, we can conclude that the latter inequality is equivalent to

$$\|\theta_i\|_{E_i(\hat{\theta}_i^*)} \leq \rho^{-1}\left(\left(8\Delta_\eta D_\theta (b-a) D_f^2 L^2\right)^{1/4}\right). \tag{4.46}$$

Thus, we have just shown that there exists a point x', y' in the state space of system 4.4 to 4.6 and parameters γ, ε, such that for all $s_{i,0} \in \Omega_s$ and every $\theta_i \in [\theta_{\min}, \theta_{\max}]$ the estimate $\hat{\theta}_i(x_i(t, x', y'))$ converges into a small neighborhood of $E_i(\theta_i)$ in finite time and stays there for an arbitrarily long time. The size of this neighborhood can be characterized by a $\mathcal{K}_\infty$-function of Δ_η, when Δ_η is sufficiently small. Let us now show that this convergence is uniform with respect to θ_i.

2. Uniformity. Consider equation 4.29. According to equations 4.23 and 4.29, trajectories passing through a point (x', y') satisfying equation 4.18 at $t = 0$ also satisfy the following constraint,

$$\begin{aligned}\exists k' \in \mathbb{Z}: \ \ h(0) - h(\infty) &= \gamma \int_0^\infty \|e_i(\tau, e_i(0), \theta_i, \eta_i(\tau))\|_\varepsilon d\tau \\ &\leq \pi - \nu_x + 2\pi k' < \infty,\end{aligned} \tag{4.47}$$

for all $\theta_i \in [\theta_{\min}, \theta_{\max}]$ and $e_i(0)$. We will use this property to demonstrate that there is a point (x', y'), $\sqrt{x'^2 + y'^2} = 1$, $\|\hat{\theta}_i(x')\|_{E_i(\theta_i)} \geq \Delta_0$, $\Delta_0 \in \mathbb{R}_{>0}$ such that for any $\theta_i \in [\theta_{\min}, \theta_{\max}]$, the estimate $\hat{\theta}_i(x_i(t, x', y'))$ converges into a set

$$\|\theta_i\|_{E_i(\hat{\theta}_i)} \leq \rho^{-1}\left(\left(8\Delta_\eta D_\theta (b-a) D_f^2 L^2\right)^{1/4}\right) \tag{4.48}$$

in finite time $T'(\theta_i)$ for all t_0, $\hat{s}_{i,0} \in \Omega_s$, and stays there for all $t \geq t_0 + T'(\theta_i)$. Furthermore, the value of $T'(\theta_i)$ is bounded from above for all $\theta_i \in$

$[\theta_{\min}, \theta_{\max}]$. In other words, there exists $T'_{\max} \in \mathbb{R}_{>0}$:

$$T'(\theta_i) \leq T'_{\max} \ \forall \ \theta_i \in [\theta_{\min}, \theta_{\max}]. \tag{4.49}$$

The fact that estimate $\hat{\theta}_i$ converges into a set specified by equation 4.48 in finite time $T'(\theta_i)$ and stays there for $t \geq t_0 + T'(\theta_i)$ for all $x', \ y' : \sqrt{x'^2 + y'^2} = 1$ follows immediately from equation 4.46. We must show, however, that equation 4.49 holds.

According to equations 3.4 and 4.7, there is a point $\theta_0 \in [a, b]$ such that $\|\theta_0\|_{E_i(\theta_i)} \geq \Delta_\theta$ for every $\theta_i \in \Omega_\theta$. Hence, there exists a point $\theta_{i,1} \in [a, b]$ such that

$$\inf_{\bar{\theta}_i \in E_i(\theta_i) \cap [a,b]} \|\bar{\theta}_i - \theta_{i,1}\| = \Delta_\theta.$$

Without loss of generality, suppose that the set $\Omega_1 = \{\bar{\theta}_i \in E_i(\theta_i) \cap [a, b] | \ \theta_{i,1} > \bar{\theta}_i\}$ is not empty.[9] By symbol $\theta_{i,\max}$ we denote $\theta_{i,\max} = \sup\{\Omega_1\}$. Let us pick a point $\theta_{i,2} \in [a, b]$ according to the following constraints,

$$\begin{aligned} |\theta_{i,2} - \theta_{i,1}| = |\theta_{i,2} - \theta_{i,\max}| = \Delta_\theta/2, \\ \theta_{i,1} > \theta_{i,2} > \theta_{i,\max}, \end{aligned} \tag{4.50}$$

and choose the value of ν_x in equation 4.19 such that

$$\theta_{i,2} = a + \frac{b-a}{2}(\cos(\nu_x) + 1), \ \ \nu_x \in [0, \pi].$$

According to equation 4.21, there exist $\bar{h}(\theta_{i,\max})$, k such that

$$\theta_{i,\max} = a + \frac{b-a}{2}(\cos(\bar{h}(\theta_{i,\max}) + 2\pi k) + 1), \ \ \bar{h}(\theta_{i,\max}) \in [0, \pi], \ \ k \in \mathbb{N}.$$

Given that $\theta_{i,2} > \theta_{i,\max}$, we set the value of $k = 0$ and choose $\bar{h}(\theta_{i,\max})$ in accordance with the following inequality:

$$\nu_x < \bar{h}(\theta_{i,\max}). \tag{4.51}$$

Because $|\hat{\theta}_i(\cos(\nu_x)) - \hat{\theta}_i(\cos(\nu'_x))| \leq \frac{b-a}{2}|\nu_x - \nu'_x|$ for all $\nu_x, \nu'_x \in \mathbb{R}$, conditions 4.50 and 4.51 ensure the existence of a constant $\nu'_x \leq \bar{h}(\theta_{i,\max})$, $\nu'_x = \nu_x + \Delta_\theta/(2(b-a))$ such that

$$|\hat{\theta}_i(\cos(\nu_x)) - \hat{\theta}_i(\cos(\nu''_x))| \leq \Delta_\theta/4 \ \ \forall \ \nu''_x \in [\nu_x, \nu'_x]. \tag{4.52}$$

[9] If Ω_1 is empty, then $\Omega_2 = \{\bar{\theta}_i \in E_i(\theta_i) \cap [a, b] | \ \theta_{i,1} < \bar{\theta}_i\}$ is not empty. We can proceed with the same argument, replacing interval $[0, \pi]$ with $[\pi, 2\pi]$ and sup with inf when appropriate.

Hence,

$$\|\hat{\theta}_i(\cos(\nu_x''))\|_{E_i(\theta_i)} \geq \frac{\Delta_\theta}{4} \;\; \forall \, \nu_x'' \in [\nu_x, \nu_x'].$$

The inequality above implies that the values of $\hat{\theta}_i(\cos(\nu_x''))$ are outside the $\Delta_\theta/4$-neighborhood of $E_i(\theta_i)$ for all $\nu_x'' \in [\nu_x, \nu_x']$. Furthermore, because $\hat{\theta}_i(\cos(\cdot))$ is monotone (nonincreasing) over $[\nu_x, \bar{h}(\theta_{i,\max}))$ and $\theta_{i,2} > \theta_{i,\max}$, there are no values of $\nu_x'' \in [\nu_x, \bar{h}(\theta_{i,\max}))$ such that $\|\hat{\theta}_i(\cos(\nu_x''))\|_{E_i(\theta_i)} = 0$.

Let us consider those solutions of system 4.4 to 4.6 that are passing through the following point $x_i(0) = \cos(\nu_x)$, $y_i(0) = \sin(\nu_x)$, $\hat{s}_i(0) \in \Omega_s$. Suppose that $0 < \gamma < \gamma^*$, and γ^* satisfies equation 4.27 with $h^* = \Delta_\theta/(2(b-a))$. Then, according to Tyukin et al. (2008), the sum $\nu_x + \gamma \int_0^t \|e_i(\tau)\|_\varepsilon d\tau$ converges to a point in $[\nu_x, \bar{h}(\theta_{i,\max})]$. Taking monotonicity and continuity of function $\hat{\theta}_i(\cos(\nu_x''))$ for $\nu_x'' \in [\nu_x, \bar{h}(\theta_{i,\max})]$ into account, we can conclude that trajectory $\hat{\theta}_i(x_i(t, x'(\theta_i)))$ enters the ε^*-neighborhood of $\theta_{i,\max}$ only once for all $t \in [0, \infty]$.

Let us show that the amount of time required for the system to enter this neighborhood is bounded from above for all $\theta_i \in \Omega_\theta$. Given that trajectory $\hat{\theta}_i(x_i(t, x', y'))$ enters the ε^*-neighborhood of $\theta_{i,\max}$ only once, we shall show that the amount of time the system spends outside this neighborhood is bounded from above for all $\theta_i \in \Omega_\theta$. We prove this by contradiction. Suppose that for any fixed $T_0' \in \mathbb{R}_{>0}$, there is a $\theta_i \in [\theta_{\min}, \theta_{\max}]$ such that $T'(\theta_i) \geq T_0'$. Consider the dynamics of system 4.4 to 4.6 when $s(t) = s_i(t, s_{i,0}, \theta_i, \eta_i(t))$. Let us pick a sequence of time instances $\{t_j\}_{j=1}^{\infty}$ such that $t_{j+1} - t_j = D_T$ and $D_T \geq L^*$. For each interval $[t_j, t_{j+1}]$, we consider two alternative possibilities:

1. The norm $\|\hat{\theta}_i(t_j) - \hat{\theta}_i(\tau)\|_{\infty,[t_j,t_{j+1}]} \leq \epsilon$, $\epsilon \in \mathbb{R}_{>0}$, $\epsilon \leq D_\theta^{-1}\Delta_\eta$.
2. The norm $\|\hat{\theta}_i(t_j) - \hat{\theta}_i(\tau)\|_{\infty,[t_j,t_{j+1}]} > \epsilon$.

In case the first alternative applies, according to equation 4.44, the following estimate holds: $\int_{t_j}^{t_{j+1}} \|e_i(\tau)\|_\varepsilon d\tau \geq \delta^*$. Hence $h(t_j) - h(t_{j+1}) > \gamma\delta^*$. When the second alternative holds, $\|\hat{\theta}_i(t_i) - \hat{\theta}_i(\tau)\|_{\infty,[t_j,t_{j+1}]} > \epsilon$, we can conclude, using inequality 4.22, that

$$\|\gamma \int_{t_j}^{\tau} \|e_i(\tau_1)\|_\varepsilon d\tau_1\|_{\infty,[t_j,t_{j+1}]} > \epsilon \frac{2}{b-a}.$$

Given that $h(t)$ is monotone with respect to t, we obtain that $h(t_j) - h(t_{j+1}) > \epsilon 2/(b-a)$. Thus, we have shown that

$$h(t_j) - h(t_{j+1}) > \min\{\gamma\delta^*, \epsilon 2/(b-a)\} = \Delta_h$$

for all j such that $\left\|\hat{\theta}_i(\tau)\right\|_{E_i(\theta_i)} \geq \varepsilon^*$ for all $\tau \in [t_j, t_{j+1}]$. Given that $h(t)$ is nonincreasing and T' is arbitrarily large, there would be a time instant $t_m \leq$

T' when $\sum_j^m h(t_j) - h(t_{j+1}) \geq m\Delta_h > \pi - \nu_x + 2\pi k'$. This, however, would contradict equation 4.47. Hence property 4.49 is proven.

3. Structurally stable prototype. So far we have shown that for the given system 4.4 to 4.6, there exists a nonempty set of parameters γ, ε, and $x', y' : \sqrt{x'^2 + y'^2} = 1$, such that trajectories $x_i(t, x', y')$, $y_i(t, x', y')$ converge to a point on the unit circle in $\mathbb{R}^2$, and variable $\hat{\theta}_i(x_i(t, x', y'))$ reaches a given small neighborhood of $E_i(\theta_i)$ (see equation 4.48) within finite time $T'_{\max}$ for all $\theta_i \in [\theta_{\min}, \theta_{\max}]$.

Let us now consider perturbed system 4.9, 4.10 where $\delta \in \mathbb{R}_{>0}$ and initial conditions are selected in a neighborhood of x', y':

$$(x_i(0), y_i(0)) \in \Omega(x', y') = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x - x')^2 + (y - y')^2} \leq \delta_r\}, \ \ \delta_r \in \mathbb{R}_{>0}. \tag{4.53}$$

In order to distinguish solutions of 4.9, 4.10 from the solutions of unperturbed system 4.4 to 4.6, we denote the latter by symbols $x_i^*(t, x_i(0), y_i(0))$, $y_i^*(t, x_i(0), y_i(0))$, and $\hat{s}_i^*(t, \theta_i, s_{i,0}, \eta_i(t))$. For notational compactness, we also denote the state vector of the ith subsystem of 4.4 to 4.6 as $\mathbf{q}_i^* = (\hat{s}_i^*, x_i^*, y_i^*)$ and the state vector of the ith subsystem of 4.9, 4.10 as $\mathbf{q}_i$.

Solutions of system 4.9, 4.10 are bounded:

$$\begin{aligned}
\|\hat{s}_i(t, \hat{s}_{i,0}, \eta_i(t))\|_{\infty,[0,\infty]} &\leq |\hat{s}_{i,0}| + (\max\{|a|, |b|\} D_\theta + \Delta_\eta)/\varphi_{\min}, \\
\|x_i(t, x_i(0), y_i(0))\|_{\infty,[0,\infty]} &\leq \max\{1, \sqrt{x_i(0)^2 + y_i(0)^2}\}, \\
\|y_i(t, x_i(0), y_i(0))\|_{\infty,[0,\infty]} &\leq \max\{1, \sqrt{x_i(0)^2 + y_i(0)^2}\}.
\end{aligned} \tag{4.54}$$

Hence, for all $\hat{s}_i(0), x_i(0), y_i(0) \in \Omega_s \times \Omega(x', y')$ there exists a constant D_0 such that $\|\mathbf{q}_i(t)\|_{\infty,[0,\infty]} \leq D_0$ for all θ_i. Let us rewrite equation 4.9, 4.10, as follows:

$$\begin{aligned}
\dot{\hat{s}}_i &= -\varphi_i(\hat{s}_i) + f_i(\xi(t), \hat{\theta}_i(x_i)) \\
\dot{x}_i &= \gamma \|\hat{s}_i - s\|_\varepsilon \left(x_i - y_i - x_i(x_i^2 + y_i^2)\right) + \gamma\delta \cdot \varepsilon_x(x_i, y_i) \\
\dot{y}_i &= \gamma \|\hat{s}_i - s\|_\varepsilon (x_i + y_i - y_i\left(x_i^2 + y_i^2\right)) + \gamma\delta \cdot \varepsilon_y\left(x_i, y_i\right),
\end{aligned} \tag{4.55}$$

where

$$\begin{aligned}
\varepsilon_x(x_i(t), y_i(t)) &= x_i(t) - y_i(t) - x_i(t)\left(x_i^2(t) + y_i^2(t)\right); \\
\varepsilon_y(x_i(t), y_i(t)) &= x_i(t) + y_i(t) - y_i(t)\left(x_i^2(t) + y_i^2(t)\right)
\end{aligned}$$

The right-hand side of equations 4.4 to 4.6 is locally Lipschitz in $\hat{s}_i, x_i, y_i$ (and so is the right-hand side of equations 4.9 and 4.10). We denote its corresponding Lipschitz constant in the domain specified by equation 4.54 by symbol $L_i(D_0)$. Furthermore, provided that equation 4.54 holds, $\varepsilon_x(x_i(t), y_i(t))$, $\varepsilon_y(x_i(t), y_i(t))$ are globally bounded with respect to t. Let us denote this bound by symbol B:

$$\max\left\{\|\varepsilon_x(x_i(t), y_i(t))\|_{\infty,[0,\infty]}, \|\varepsilon_y(x_i(t), y_i(t))\|_{\infty,[0,\infty]}\right\} = B.$$

For the notational compactness, let us rewrite equation 4.55 as follows:

$$\dot{\mathbf{q}}_i = \mathbf{f}(\mathbf{q}_i, s(t), \xi(t)) + \gamma\delta \cdot \mathbf{g}(\mathbf{q}_i), \tag{4.56}$$

where $\mathbf{f}(\mathbf{q}_i, s(t), \xi(t))$ and $\mathbf{g}(\mathbf{q}_i)$ are defined to copy the right-hand side of equation 4.55. Notice that $\|\mathbf{f}(\mathbf{q}_i, s(t), \xi(t))\| \leq L_i(D_0)\|\mathbf{q}_i\|$, $\|\mathbf{g}(\mathbf{q}_i)\| \leq B\sqrt{2}$.

According to the theorem on continuous dependence of solutions of an ordinary differential equation on parameters and initial conditions (see, e.g., Khalil, 2002, theorem 3.4) the following holds:

$$\begin{aligned}\|\mathbf{q}_i(t) - \mathbf{q}_i^*(t)\| \leq{} & \|\mathbf{q}_i(t_0) - \mathbf{q}_i^*(t_0))\|e^{L_i(D_0)(t-t_0)} \\ & + \frac{\delta\gamma B\sqrt{2}}{L_i(D_0)}\left(e^{L_i(D_0)(t-t_0)} - 1\right).\end{aligned} \tag{4.57}$$

When the values of $\hat{s}_{i,0}$ and $\hat{s}_{i,0}^*$ coincide, estimate 4.57 implies that

$$\|\mathbf{q}_i(t) - \mathbf{q}_i^*(t)\| \leq \delta_r e^{L_i(D_0)(t-t_0)} + \frac{\delta\gamma B\sqrt{2}}{L_i(D_0)}\left(e^{L_i(D_0)(t-t_0)} - 1\right). \tag{4.58}$$

This ensures the existence of $\delta_r \in \mathbb{R}_{>0}$, $\delta \in \mathbb{R}_{>0}$ such that for a fixed, yet arbitrarily large, time $T''(\delta_r, \delta) > T'_{\max}$, solutions of system 4.9, 4.10 passing through a point from $\Omega(x', y')$ at $t = t_0$ will remain within a fixed, yet arbitrarily small, neighborhood of a solution of system 4.4 to 4.6 with initial conditions $x_i(t_0) = x'$, $y_i(t_0) = y'$. The value of $T'_{\max}$ does not depend on δ_r, δ.

Taking equation 4.20 into account, we can conclude that the set $x_i^2 + y_i^2 = 1$ is globally attracting in the state space of system 4.9, 4.10 for almost all initial conditions (except when $x_i(t_0) = 0$, $y_i(t_0) = 0$). This implies that solutions starting in $\Omega(x', y')$ will remain there. In addition, according to equation 4.19, for any $t_0 \geq 0$ a δ_r-vicinity of (x', y') will be visited within at least time $t' \leq t_0 + 2\pi/(\gamma \cdot \delta)$. Hence, we have just shown that for all $t_0 \geq 0$, solutions starting at $\Omega_s \times \Omega(x', y')$ approach the target set within a fixed time $T'_{\max}$ and stay in its vicinity for an arbitrarily long time $T''(\delta_r, \delta)$. The latter time is a function of δ_r, δ: the smaller the values of δ_r, δ, the larger the value of $T''(\delta_r, \delta)$.

4. Realizability. Let us finally show that system 4.9, 4.10 can be realized by a recurrent neural network. More precisely, we wish to prove that there exists a system 3.8 such that $\mathbf{x} = \boldsymbol{\zeta}_{\mathbf{1}} \oplus \boldsymbol{\zeta}_{\mathbf{2}} \oplus \cdots \oplus \boldsymbol{\zeta}_{\mathbf{N_f}}$, $\boldsymbol{\zeta}_i \in \mathbb{R}^3$, $\boldsymbol{\zeta}_i = \zeta_{i,1} \oplus \zeta_{i,2} \oplus \zeta_{i,3}$, $i = \{1, \ldots, N_f\}$ and solutions $\boldsymbol{\zeta}_i(t, \mathbf{q}_{i,0})$ are sufficiently close to $\mathbf{q}_i(t, \mathbf{q}_{i,0})$, where $\mathbf{q}_{i,0} \in \Omega_s \times \Omega(x', y') \subset \mathbb{R}^3$.

It is clear that the right-hand side of system 4.9, 4.10 is a continuous and locally Lipschitz function. To proceed further, we use the following result by Cybenko (1989):

Theorem 2 (Cybenko, 1989). *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any continuous sigmoid-type function. Then finite sums of the form*

$$G(\boldsymbol{\zeta}) = \sum_{m=1}^{N} \alpha_m \sigma(\omega_m^T \boldsymbol{\zeta} + \beta_m),\ \ \boldsymbol{\zeta} \in \mathbb{R}^n,\ \ \boldsymbol{\omega}_m \in \mathbb{R}^n,\ \ \alpha_m,\ \ \beta_m \in \mathbb{R}$$

are dense in $\mathcal{C}^0([0, 1]^n)$.[10]

According to theorem 2, for any arbitrarily small $\varepsilon_N \in \mathbb{R}_{>0}$, any given bounded intervals $\Omega_x \subset \mathbb{R}$, $\Omega_y \subset \mathbb{R}$, and any

$$s(t), \xi(t):\ \max\{\|s(t)\|_{\infty,[0,\infty]}, \|\xi(t)\|_{\infty,[0,\infty]}\} < M,\ \ M \in \mathbb{R}_{>0},$$

there exist $N \in \mathbb{N}$, $\omega_{j,m} \in \mathbb{R}^5$, $\alpha_{j,m} \in \mathbb{R}$, $\beta_{j,m} \in \mathbb{R}$, $j = 1, 2, \ldots, N$ such that

$$\left|\sum_{m=1}^{N} \alpha_{j,m}\sigma(\omega_{j,m}^T \cdot \mathbf{u}(s(t), \xi(t), \boldsymbol{\zeta}_i) + \beta_{j,m}) - \mathbf{f}_j(\boldsymbol{\zeta}_i, s(t), \xi(t)) - \gamma\delta \cdot \mathbf{g}_j(\boldsymbol{\zeta}_i)\right| < \frac{\varepsilon_N}{3},$$
$$\mathbf{u}(s(t), \xi(t), \boldsymbol{\zeta}_i) = s(t) \oplus \xi(t) \oplus \boldsymbol{\zeta}_i, \tag{4.59}$$

where $\boldsymbol{\zeta}_i \in \Omega_s \times \Omega_x \times \Omega_y$, and $\mathbf{f}_j(\boldsymbol{\zeta}_i, s(t), \xi(t))$, $\mathbf{g}_j(\boldsymbol{\zeta}_i)$, $j = 1, 2, 3$ denote the jth components of the vector fields $\mathbf{f}(\boldsymbol{\zeta}_i, s(t), \xi(t))$, $\mathbf{g}(\boldsymbol{\zeta}_i)$, respectively. It follows from equation 4.59 that there exist N, $\boldsymbol{\omega}_{j,m}$, $\alpha_{j,m}$, $\beta_{j,m}$ such that

$$\sum_{m=1}^{N} \alpha_{j,m}\sigma\left(\boldsymbol{\omega}_{j,m}^T \cdot \mathbf{u}(s(t), \xi(t), \boldsymbol{\zeta}_i) + \beta_{j,m}\right) = \mathbf{f}_j(\boldsymbol{\zeta}_i, s(t), \xi(t)) + \gamma\delta \cdot \mathbf{g}_j(\boldsymbol{\zeta}_i)$$
$$+\Delta_j(\boldsymbol{\zeta}_i, s(t), \xi(t)),$$
$$\mathbf{u}(s(t), \xi(t), \boldsymbol{\zeta}_i) = s(t) \oplus \xi(t) \oplus \boldsymbol{\zeta}_i, \tag{4.60}$$

[10]Symbol $\mathcal{C}^0([0, 1]^n)$ denotes the space of all continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$ defined on the hypercube $[0, 1]^n \subset \mathbb{R}^n$.

where $\Delta_j(\boldsymbol{\zeta}_i, s(t), \xi(t))$ are continuous and

$$|\Delta_j(\boldsymbol{\zeta}_i, s(t), \xi(t))| < \frac{\varepsilon_N}{3} \;\; \forall\, j = 1, 2, 3.$$

Let us choose $\Omega_x = [-v, v]$, $\Omega_y = [-v, v]$ where $v \in \mathbb{R}_{>0}$, $v > 1$ and consider the dynamics of

$$\begin{aligned}
\dot{\boldsymbol{\zeta}}_i &= \mathbf{f}(\boldsymbol{\zeta}_i, s(t), \xi(t)) + \gamma\delta \cdot \mathbf{g}(\boldsymbol{\zeta}_i) + \Delta(\boldsymbol{\zeta}_i, s(t), \xi(t)),\\
\Delta(\boldsymbol{\zeta}_i, s(t), \xi(t)) &= \Delta_1(\boldsymbol{\zeta}_i, s(t), \xi(t)) \oplus \Delta_2(\boldsymbol{\zeta}_i, s(t), \xi(t)) \oplus \Delta_3(\boldsymbol{\zeta}_i, s(t), \xi(t)),\\
&\|\Delta(\boldsymbol{\zeta}_i, s(t), \xi(t))\| \le \varepsilon_N.
\end{aligned} \tag{4.61}$$

System 4.61 has a globally attracting invariant set (for almost all initial conditions), which can be characterized as

$$\{\boldsymbol{\zeta}_i \in \mathbb{R}^3 |\, 1 - \rho(\varepsilon_N) \le \zeta_{i,2}^2 + \zeta_{i,3}^2 \le 1 + \rho(\varepsilon_N)\},\;\; \rho \in \mathcal{K}_\infty.$$

This follows immediately from the fact that equation 4.56 is structurally stable and has a globally attracting invariant set (for almost all initial conditions). Furthermore, for any given ε_N and a bounded set of initial conditions $\Omega_\zeta(r) = \{\boldsymbol{\zeta}_i \in \mathbb{R}^3 |\; \|\boldsymbol{\zeta}_i\| \le r,\;\; r \in \mathbb{R}_{>0}\}$, there exists a constant B_1 such that $\|\boldsymbol{\zeta}_i(t)\|_{\infty,[0,\infty]} < B_1$. Hence, solutions of system

$$\begin{aligned}
\dot{\zeta}_{i,j} &= \sum_{m=1}^{N} \alpha_{j,m}\sigma(\boldsymbol{\omega}_{j,m}^T \cdot \mathbf{u}(s(t), \xi(t), \boldsymbol{\zeta}_i) + \beta_{j,m}),\\
\mathbf{u}&(s(t), \xi(t), \boldsymbol{\zeta}_i) = s(t) \oplus \xi(t) \oplus \boldsymbol{\zeta}_i,\;\; j = 1, 2, 3
\end{aligned} \tag{4.62}$$

are bounded for all initial conditions from $\Omega_\zeta(r)$ provided that inequality 4.59 holds over sufficiently large intervals Ω_x, Ω_y (for sufficiently large v). Furthermore, given that ε_N is sufficiently small, solutions of equation 4.62 enter domain $\Omega_s \times \Omega(x', y')$ specified by equation 4.53 in finite time. Finally, according to equality 4.60 and theorem 3.4 in Khalil (2002), solutions of equation 4.62 starting in $\Omega(x', y')$, satisfy the following inequality:

$$\begin{aligned}
\|\mathbf{q}_i(t, \mathbf{q}_{i,0}) - \boldsymbol{\zeta}_i(t, \mathbf{q}_{i,0})\| &\le \frac{\varepsilon_N}{L_i(D_0)} \left(e^{L_i(D_0)(t-t_0)} - 1\right),\\
\mathbf{q}_{i,0} &\in \Omega_s \times \Omega(x', y').
\end{aligned} \tag{4.63}$$

Hence, for any $t \ge 0$, solutions of equation 4.62 starting from $\Omega_\zeta(r)$ approach the target set within a fixed time (dependent on δ) and stay in its vicinity arbitrarily long provided that the values δ in equation 4.55 and ε_N in 4.59 to 4.61 are sufficiently small. The possibility of the latter follows from theorem 2: the value of ε_N can be made arbitrarily small by the appropriate choice of parameters N, $\boldsymbol{\omega}_{j,m}$, $\alpha_{j,m}$, $\beta_{j,m}$, and the value of δ can be made arbitrarily small because it is our design parameter.

Taking equations 4.63, 4.58, 4.12, and 4.9 into account, we conclude the proof by choosing $h_{f,i}(\mathbf{x}, s)$, $h_{\theta,i}(\mathbf{x}, s)$ as follows:

$$\begin{aligned} h_{f,i}(\mathbf{x}, s) &= h_{f,i}(\boldsymbol{\zeta}_1 \oplus \cdots \oplus \boldsymbol{\zeta}_{N_f}, s) = s - \zeta_{i,1}, \\ h_{\theta,i}(\mathbf{x}, s) &= h_{\theta,i}(\boldsymbol{\zeta}_1 \oplus \cdots \oplus \boldsymbol{\zeta}_{N_f}, s) = a + \frac{b-a}{2}(\zeta_{i,2} + 1). \end{aligned} \tag{4.64}$$

The theorem is proven.

Before concluding this section, we provide several remarks regarding theorem 1.

Remark 1 (readout from the outputs). As follows from the theorem, the class to which a given signal belongs can be determined from the values of $h_{f,j}(\mathbf{x}(t), s(t))$, $j = \{1, \ldots, N_f\}$ (specified, for example, by equation 4.64) within a finite interval of time. When $s(t) = s_i(t, s_{i,0}, \theta_i, \eta_i(t))$ the values of $h_{f,i}(\mathbf{x}(t), s(t))$ should approach a small neighborhood of zero and stay there for a sufficiently long time. The estimate of θ_i up to its equivalence class is available from the values of $h_{\theta,i}(\mathbf{x}(t), s(t))$ over the same interval. As follows from our proof, the more accurately the RNN approximates system 4.55, the larger the interval of time when $h_{f,i}(\mathbf{x}(t), s(t))$ is in the vicinity of zero. Indeed, if the approximation error is small (e.g., the value of ε_N in equation 4.59 is small), then for a given $\delta^* \in \mathbb{R}_{>0}$ the right-hand side of equation 4.63 should not exceed δ^* for all $t \in [t_0 + T]$:

$$0 \leq T = \frac{1}{L_i(D_0)} \ln\left(\frac{\delta^* L_i(D_0)}{\varepsilon_N} + 1\right).$$

The smaller the ε_N, the larger the value of T, and hence the longer the interval of time when trajectories of the RNN stay within the δ^*-neighborhood of the solutions of equation 4.55. The latter, as follows from equation 4.58, can be made arbitrarily close to that of the converging prototype 4.4 to 4.6 by choosing the value of δ sufficiently small. Thus, the function $h_{f,i}(\mathbf{x}(t), s(t))$ as defined by equation 4.64, will asymptotically approach zero and stay in its small neighborhood for a sufficiently long time, subject to the choice of δ and ε_N.

From a practical viewpoint, it might sometimes be preferable to read out from the RNN outputs directly rather than having to satisfy ourselves with the existence of two sets of readout functions, for state and input, respectively, of the RNN. Although this option is not stated explicitly in theorem 1, it can easily be shown that the preferred option can indeed be realized.

Adding to recurrent subsystem 3.8, a feedforward part realizing continuous "output" functions 4.64 enables direct readout from the RNN outputs.

Remark 2 (convergence to an attractor). Theorem 1 does not imply that recognition of the class of the input signal $s(t)$ involves convergence of the RNN state to an attractor. Yet its formulation does not exclude this option either. In fact, when $f_i(\xi(t), \theta_i)$ satisfies some additional restrictions (e.g., linear or monotone parameterization with respect to θ_i), it is possible to replace equations 4.5 and 4.6 with another prototype system: one that converges to a point attractor exponentially (Tyukin, Prokhorov, & van Leeuwen, 2007). This implies that it depends substantially on the properties of $f_i(\xi(t), \theta_i)$ whether the network state will behave intermittently or asymptotically converge to an attractor. It is important, however, that in both cases, an RNN will successfully solve the recognition problem.

Remark 3 (multidimensional uncertainty). Although the theorem applies to the case where θ_i is a scalar, it can be trivially extended to the case where uncertain parameters are vectors from a bounded domain $\Omega_{\theta,d} \subset \mathbb{R}^d$. To do so one, needs to find a Lipschitz mapping $\boldsymbol{\lambda} : \mathbb{R} \to \mathbb{R}^d$ such that for a given small $\varepsilon_\lambda \in \mathbb{R}_{>0}$, the following property holds:

$$\forall\, \boldsymbol{\theta}_i \in \Omega_{\theta,d}\ \exists\, \theta_i \in \Omega_\theta :\ \|\boldsymbol{\theta}_i - \lambda(\theta_i)\| < \varepsilon_\lambda.$$

Hence, the problem will reduce to the scalar case to which theorem 1 applies.

Remark 4 (number of dynamic states). Our proof of theorem 1 not only demonstrates the possibility for an RNN to classify uncertain functions of time adaptively. It also allows us to estimate the number of dynamic nodes (e.g., the value of N_x in equation 3.8), that is sufficient for successful classification. As follows immediately from equations 4.55, 4.61, and 4.62, the number of the dynamic states, N_x, can be as small as

$$N_x = 3N_f, \tag{4.65}$$

where N_f is the number of functions $f_i(\cdot,\cdot)$ to be classified. Further, suppose that N is the number of sigmoidal terms ensuring sufficiently accurate approximation of each function on the right-hand side of equation 4.55. Then the total number of sigmoidal functions in the network, N_{total} can, in principle, be estimated as follows:

$$N_{\text{total}} = 3N_f N. \tag{4.66}$$

These estimates, although inheriting linear dependence on N_f, are still somewhat conservative. Simple numerical examples show that there is room for further improvements. Consider, for instance, the following set of signals:

$$\dot{s}_i = -s_i + f_i(\xi(t), \theta_i),\ i = \{1, 2\},\\ \theta_i \in [0, 4\pi], \tag{4.67}$$

where

$$\begin{aligned} f_1(\xi(t), \theta_1) &= \sin(\xi(t)\theta_1) + \cos(\xi(t)\theta_1)\\ f_2(\xi(t), \theta_i) &= \sin(\xi(t)\theta_2) + \cos^3(\xi(t)\theta_2)\\ \xi(t) &= \sin(t). \end{aligned} \tag{4.68}$$

According to equations 4.65 and 4.66, the number of dynamic states in an RNN that adaptively classifies signals 4.67 and 4.68 could be as small as six. Assuming that only two sigmoid functions are used to approximate the right-hand side of each equation of 4.55, the total number of the sigmoidal nodes in the RNN is 12. In our numerical simulations, we have found, however, that an RNN with 10 recurrent nodes already is able to solve the adaptive classification problem of signals $s_i(t)$ defined by equations 4.67 and 4.68. Results of this experiment are provided in Figure 4.

In this example we used a network with 10 recurrent nodes. This number obviously exceeds the value provided by estimate 4.65. On the other hand, adaptive classification is achieved by an RNN with a smaller total number of sigmoidal nodes than that predicted by equation 4.66. These results motivate further attention to this topic.

5 Conclusion

We provided a formal proof that an RNN with fixed weights can serve as a universal adaptive classifier of both static and dynamic inputs. The number of dynamical states in an RNN recognizing N_f different classes of signals $s_i(t)$ can, according to our analysis, be as small as $3N_f$; it grows linearly with the size of the set of signals to be classified.

We stated the classification and recognition problems in a behavioral context in which, over time, the desired input-output relationship is achieved. Finding a solution corresponds to a network dynamics in which the state reaches a given neighborhood of the a priori specified set and stays there for a sufficiently long time, provided that input to the network belongs to a given class (problem 1). With these ramifications, RNNs solve the problem of adaptively classifying time-dependent signals. We did not set out to guarantee, however, that the state of the RNN will asymptotically converge to an equilibrium or its small vicinity as a result of recognition. On the other

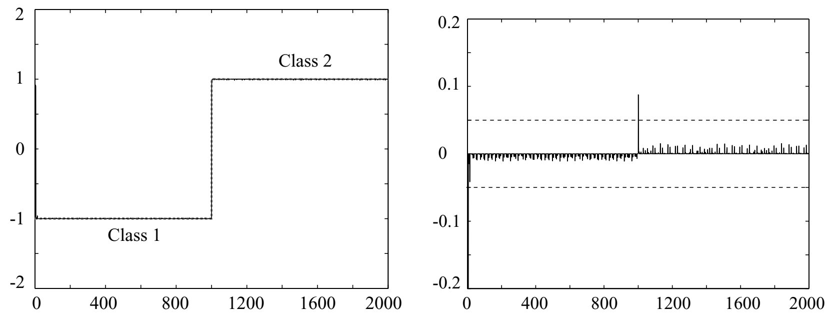

Figure 4: Adaptive classification of signals 4.65 and 4.66 by an RNN with 10 dynamic states. The network had two inputs, one for signals $s_i(t)$ and the other for signal $\xi(t)$, and one output. When $s_1(t)$ is present on the first input, the output should converge to -1; when signal $s_2(t)$ is present, the output should converge to 1. (a) Trajectories of the actual output (black solid line) and desired performance (gray dashed line) of the network as functions of model time. We started with signal $s_1(t)$, in which the value of θ_1 was set to $\theta_1 = 2\pi$. In the middle of the simulation, we replaced $s_1(t)$ with $s_2(t)$, in which the value of θ_2 was set to $\theta_2 = 3\pi$. Although the parameters and signals change, the network clearly solves the classification problem correctly. The same happens for other values of $\theta_i \in [0, 4\pi]$ for which the network was trained. (b) The difference between the actual and desired responses of the network. Always after a short period of transient behavior, the error settles well within the standard 5% zone marked by the two dashed lines.

hand, the amount of time a network would spend in the vicinity of a target set can be made sufficiently large to qualify as a practical solution to the classification problem. For classification, after all, asymptotic convergence is not actually needed.

In physics and nonlinear dynamics, the phenomenon that the state of a system reaches a neighborhood of a set and stays there sufficiently long, yet inevitably escaping, only to get caught again, is called (chaotic) itinerancy Kaneko, & Tsuda (2003); the set is referred to as an attractor ruin. These descriptive concepts are currently recognized as a possible mathematical basis for modeling brain activity (Tsuda, 1991; Tsuda, & Fujii, 2004). We envisage that our current result supports this idea, by showing its considerable power in adaptive classification.

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