INTEGRALS RELATED TO

HEAT CONDUCTION AND DIFFUSION

BY

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1.Table of Integrals

This table of integrals presents the results of the computations carried out in Chapter 3. Chapter 2 presents the main results of Chapter 3 in handbook format.

Fundamental Integrals

	Integral	Ref.
(1.1)	$F(x)=\int_0^x \frac{\mathrm{erf}(w)}{w}dw$, x≥0	**(2.1)**
(1.2)	$G(x)=\int_x^\infty \frac{\mathrm{erfc}(w)}{w}dw$, x≥0	**(2.1)**
(1.3)	$F(x)=\frac{\gamma}{2}+\ln(2x)+G(x)$ (γ =0.5772156649015328606…)	**(2.1)**
(1.4)	$\int_0^x e^{-a^2w^2}\ln w dw=\frac{\sqrt{\pi}}{2a}[\mathrm{erf}(ax)\ln x-F(ax)]$, x≥0	**(2.1)**
(1.5)	$\int_x^\infty e^{-a^2w^2}\ln w dw=\frac{\sqrt{\pi}}{2a}[\mathrm{erfc}(ax)\ln x+G(ax)]$, x≥0	**(2.1)**
(1.6)	$\int_0^\infty e^{-a^2w^2}\ln w dw=-\frac{\sqrt{\pi}}{2a}\left[\frac{\gamma}{2}+\ln(2a)\right].$	**(2.1)**
(1.7)	$G_\nu(x)\equiv\int_1^\infty \frac{E_\nu(xw)}{w^\nu}dw=\frac{-\partial E_\nu(x)}{\partial\nu}=\int_1^\infty \frac{e^{-xw}\ln w}{w^\nu}dw$, x>0, ν>0	**(2.2)**
(1.8)	$G_{1/2}(x)=2\sqrt{\frac{\pi}{x}}G(\sqrt{x})$, x>0	**(2.2)**
(1.9)	$\int_x^\infty \frac{E_\mu(w)}{w^\nu}dw=\frac{1}{x^{\nu-1}}\frac{E_\nu(x)-E_\mu(x)}{\mu-\nu}$ or	**(2.2)**
(1.10)	$\int_1^\infty \frac{E_\mu(xw)}{w^\nu}dw=\frac{E_\nu(x)-E_\mu(x)}{\mu-\nu}$, x>0, μ>0, ν>0, μ≠ν	**(2.2)**
(1.11)	$\int_x^\infty \frac{E_\nu(w)}{w^\nu}dw=\frac{1}{x^{\nu-1}}G_\nu(x)$, x>0 or	**(2.2)**
(1.12)	$\int_1^\infty \frac{E_\nu(xw)}{w^\nu}dw=G_\nu(x)$, x>0, ν>0 (μ = ν above)	**(2.2)**
(1.13)	$\int_1^\infty \frac{E_{1/2}(xw)}{\sqrt{w}}dw=G_{1/2}(x)=2\sqrt{\frac{\pi}{x}}G(\sqrt{x})$	**(2.2)**
(1.14)	$I_n(b,x)=\int_x^\infty \frac{e^{-b^2w^2}\ln w}{w^n}dw$, $n=0, 1, 2, \ldots$, x > 0	**(2.2)**
(1.15)	$I_n(b,x)=\frac{\ln x}{2x^{n-1}}E_{(n+1)/2}(b^2x^2)+\frac{1}{4x^{n-1}}G_{(n+1)/2}(b^2x^2)$, x>0	**(2.2)**
(1.16)	$I_0(b,x)=\int_x^\infty e^{-b^2w^2}\ln w dw=\frac{\sqrt{\pi}}{2b}[\mathrm{erfc}(bx)\ln x+G(bx)]$	**(2.2)**
(1.17)	$\frac{(n+1)}{2b^2}I_{n+2}+I_n=\frac{\ln x}{2b^2}\cdot\frac{e^{-b^2x^2}}{x^{n+1}}+\frac{1}{4b^2x^{n+1}}E_{(n+3)/2}(b^2x^2)$, $n\ge 0$	**(2.2)**

(1.18) $$J_\nu(a,x)=\int_x^\infty \frac{\mathrm{erf}(aw)}{w^\nu}dw=\frac{1}{(\nu-1)x^{\nu-1}}\left[\mathrm{erf}(ax)+\frac{ax}{\sqrt{\pi}}E_{\nu/2}(a^2x^2)\right],\qquad \nu\neq 1$$

(1.19) $$J_\nu^c(a,x)=\int_x^\infty \frac{\mathrm{erfc}(aw)}{w^\nu}dw=\frac{1}{(\nu-1)x^{\nu-1}}\left[\mathrm{erfc}(ax)-\frac{ax}{\sqrt{\pi}}E_{\nu/2}(a^2x^2)\right],\qquad \nu\neq 1$$

(1.20) $$J_2(a,x)=\int_x^\infty \frac{\mathrm{erf}(aw)}{w^2}dw=\frac{\mathrm{erf}(ax)}{x}+\frac{a}{\sqrt{\pi}}E_1(a^2x^2),\qquad \nu=2$$

(1.21) $$J_2^c(a,x)=\int_x^\infty \frac{\mathrm{erfc}(aw)}{w^2}dw=\frac{\mathrm{erfc}(ax)}{x}-\frac{a}{\sqrt{\pi}}E_1(a^2x^2),\qquad \nu=2$$

(1.22) $$J_3(a,x)=\int_x^\infty \frac{\mathrm{erf}(aw)}{w^3}dw=\frac{\mathrm{erf}(ax)}{2x^2}+\frac{a}{x}\mathrm{ierfc}(ax),\qquad \nu=3$$ **(2.11)**

(1.23) $$J_3^c(a,x)=\int_x^\infty \frac{\mathrm{erfc}(aw)}{w^3}dw=\frac{\mathrm{erfc}(ax)}{2x^2}-\frac{a}{x}\mathrm{ierfc}(ax)=\frac{2i^2\mathrm{erfc}(ax)}{x^2},\qquad \nu=3$$ **(2.11)**

(1.24) $$I_5(a,b,x)=\int_x^\infty e^{-a^2w^2}\mathrm{erfc}(bw)dw=\sum_{k=0}^\infty \frac{(1/2)_k}{k!}\left(\frac{a^2}{a^2+b^2}\right)^k E_{k+3/2}(d^2x^2),\ a\le b$$ **(2.3)**

$$=\frac{\sqrt{\pi}}{2a}\mathrm{erfc}(ax)\,\mathrm{erfc}(bx)-\frac{b}{a}I_5(b,a,x),\qquad d^2=a^2+b^2,\qquad a>b$$ **(2.3)**

(1.25) $$J_5(a,b,x)=\int_x^\infty e^{-a^2w^2}\mathrm{erf}(bw)dw=\frac{\sqrt{\pi}}{2a}\mathrm{erfc}(ax)-I_5(a,b,x)$$ **(2.3)**

(1.26) $$U_5(a,b,x)=\int_0^x e^{-a^2w^2}\mathrm{erfc}(bw)dw=\frac{1}{a\sqrt{\pi}}\tan^{-1}\frac{a}{b}-I_5(a,b,x)$$ **(2.3)**

(1.27) $$V_5(a,b,x)=\int_0^x e^{-a^2w^2}\mathrm{erf}(bw)dw=\frac{\sqrt{\pi}}{2a}\mathrm{erf}(ax)-U_5(a,b,x)$$ **(2.3)**

(1.28) $$\int_T^\infty \mathrm{erfc}(aw)\,\mathrm{erfc}(bw)dw=\frac{1}{2}\int_0^t \frac{\mathrm{erfc}(a/\sqrt{\tau})\mathrm{erfc}(b/\sqrt{\tau})}{\tau^{3/2}}d\tau,\qquad T=\frac{1}{\sqrt{t}}$$ **(2.8)**

$$=\frac{e^{-a^2T^2}\mathrm{erfc}(bT)}{a\sqrt{\pi}}+\frac{e^{-b^2T^2}\mathrm{erfc}(aT)}{b\sqrt{\pi}}-T\,\mathrm{erfc}(aT)\,\mathrm{erfc}(bT)-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\mathrm{erfc}(T\sqrt{a^2+b^2})$$

(1.29) $$\int_0^\infty \mathrm{erfc}(aw)\,\mathrm{erfc}(bw)dw=\frac{(2/\sqrt{\pi})}{a+b+\sqrt{a^2+b^2}}$$ **(2.8)**

(1.30) $$\int_0^T \mathrm{erf}(aw)\,\mathrm{erf}(bw)dw=\frac{1}{2}\int_t^\infty \frac{\mathrm{erf}(a/\sqrt{\tau})\mathrm{erf}(b/\sqrt{\tau})}{\tau^{3/2}}d\tau,\qquad T=\frac{1}{\sqrt{t}}$$ **(2.8)**

$$=T+\frac{1}{b}\mathrm{ierfc}(bT)+\frac{1}{a}\mathrm{ierfc}(aT)-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}-\int_T^\infty \mathrm{erfc}(aw)\,\mathrm{erfc}(bw)dw$$

$$=T\,\mathrm{erf}(aT)\,\mathrm{erf}(bT)+\frac{e^{-a^2T^2}\mathrm{erf}(bT)}{a\sqrt{\pi}}+\frac{e^{-b^2T^2}\mathrm{erf}(aT)}{b\sqrt{\pi}}-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\mathrm{erf}(T\sqrt{a^2+b^2})$$

(1.31) $$I_{20}(a,b,T)=\int_T^\infty e^{-a^2x^2}\mathrm{erf}(bx)\ln x\,dx,$$ **(2.4)**

(1.32) $$I_{20}^c(a,b,T)=\int_T^\infty e^{-a^2x^2}\mathrm{erfc}(bx)\ln x\,dx,$$ **(2.4)**

(1.33) $$P(a,b,T)=\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erf}(bw)}{w}dw \quad\text{and}\quad Q(a,b,T)=\int_T^\infty e^{-a^2w^2}E_1(b^2w^2)dw$$ **(2.5)**

Results in Terms of Fundamental Integrals (1.1)-(1.33), $T=1/\sqrt{t}$

(1.34) $$\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erf}(bw)}{w^2}dw=\frac{1}{2}\int_0^t e^{-a^2/u}\,\mathrm{erf}(b/\sqrt{u})/\sqrt{u}\,du=I_1(a,b,T)$$ **(2.6)**

$$=\frac{e^{-a^2T^2}\,\mathrm{erf}(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1[T^2(a^2+b^2)]-2a^2J_5(a,b,T)$$ **(2.6)**

(1.35) $$\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erfc}(bw)}{w^2}dw=\frac{1}{2}\int_0^t e^{-a^2/u}\,\mathrm{erfc}(b/\sqrt{u})/\sqrt{u}\,du=I_1^c(a,b,T)$$ **(2.6)**

$$=\frac{e^{-a^2T^2}\,\mathrm{erfc}(bT)}{T}-\frac{b}{\sqrt{\pi}}E_1[T^2(a^2+b^2)]-2a^2I_5(a,b,T)$$ **(2.6)**

(1.36) $$\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erf}(bw)}{w^3}dw=\frac{1}{2}\int_0^t e^{-a^2/u}\,\mathrm{erf}(b/\sqrt{u})du=I_{13}(a,b,T)$$ **(2.7)**

$$=\frac{e^{-a^2T^2}}{2T^2}\mathrm{erf}(bT)+\frac{b}{T}i\,\mathrm{erfc}(T\sqrt{a^2+b^2})-a^2P(a,b,T)$$

(1.37) $$\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erfc}(bw)}{w^3}dw=\frac{1}{2}\int_0^t e^{-a^2/u}\,\mathrm{erfc}(b/\sqrt{u})du=I_{13}^c(a,b,T)$$ **(2.7)**

$$=\frac{1}{2T^2}E_2(a^2T^2)-I_{13}(a,b,T)$$

(1.38) $$\int_0^T \frac{\mathrm{erf}(aw)\,\mathrm{erf}(bw)}{w}dw=\frac{1}{2}\int_t^\infty \frac{\mathrm{erf}(a/\sqrt{\tau})\,\mathrm{erf}(b/\sqrt{\tau})}{\tau}d\tau=$$ **(2.9)**

$$=F(aT)-\left[G(bT)-I_{19}^c(a,b,T)\ \right]_{T\to 0}+\left[G(bT)-I_{19}^c(a,b,T)\ \right]$$

(1.39) $$\int_T^\infty \frac{\mathrm{erfc}(aw)\,\mathrm{erfc}(bw)}{w}dw=\frac{1}{2}\int_0^t \frac{\mathrm{erfc}(a/\sqrt{\tau})\,\mathrm{erfc}(b/\sqrt{\tau})}{\tau}d\tau=I_{19}^c(a,b,T)$$ **(2.9)**

(1.40) $$\int_T^\infty \frac{\mathrm{erf}(aw)\,\mathrm{erfc}(bw)}{w}dw=\frac{1}{2}\int_0^t \frac{\mathrm{erf}(a/\sqrt{\tau})\,\mathrm{erfc}(b/\sqrt{\tau})}{\tau}d\tau=G(bT)-I_{19}^c(a,b,T)$$ **(2.9)**

(1.41) $$\int_T^\infty \frac{\mathrm{erf}(aw)\,\mathrm{erf}(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{\mathrm{erf}\left(a/\sqrt{\tau}\right)\mathrm{erf}\left(b/\sqrt{\tau}\right)}{\tau^{1/2}}du=I_6(a,b,T)$$ **(2.10)**

(1.42) $$\int_T^\infty \frac{\mathrm{erfc}(aw)\,\mathrm{erfc}(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{\mathrm{erfc}(a/\sqrt{\tau})\,\mathrm{erfc}(b/\sqrt{\tau})}{\tau^{1/2}}d\tau$$ **(2.10)**

$$=\frac{1}{T}-J_2(a,T)-J_2(b,T)+I_6(a,b,T)$$

(1.43) $$\int_T^\infty \frac{\mathrm{erf}(aw)\,\mathrm{erfc}(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{\mathrm{erf}(a/\sqrt{\tau})\,\mathrm{erfc}(b/\sqrt{\tau})}{\tau^{1/2}}d\tau$$ **(2.10)**

$$=J_2(a,T)-I_6(a,b,T)$$

(1.44) $$\int_T^\infty \frac{\mathrm{erf}(aw)\,\mathrm{erf}(bw)}{w^3}dw=\frac{1}{2}\int_0^t \mathrm{erf}(a/\sqrt{\tau})\,\mathrm{erf}(b/\sqrt{\tau})d\tau=W_3(a,b,T)$$ **(2.11)**

(1.45) $$\int_T^\infty \frac{\mathrm{erfc}(aw)\,\mathrm{erfc}(bw)}{w^3}dw=\frac{1}{2}\int_0^t \mathrm{erfc}(a/\sqrt{\tau})\,\mathrm{erfc}(b/\sqrt{\tau})d\tau=W_3^c(a,b,T)$$ **(2.11)**

(1.46) $$\int_T^\infty \frac{\operatorname{erf}(aw)\operatorname{erfc}(bw)}{w^3}dw = \frac{1}{2}\int_0^t \operatorname{erf}(a/\sqrt{\tau})\operatorname{erfc}(b/\sqrt{\tau})d\tau$$ $$= J_3(a,T) - W_3(a,b,T)$$ $$= J_3^c(b,T) - W_3^c(a,b,T)$$ **(2.11)**

(1.47) $$\int_T^\infty e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw = \frac{1}{2}\int_0^t e^{-c^2/u}\frac{\operatorname{erf}(a/\sqrt{u})\operatorname{erf}(b/\sqrt{u})}{u^{3/2}}du$$ $$= I_3(a,b,c,T)$$ **(2.12)**

(1.48) $$\int_T^\infty e^{-c^2w^2}\operatorname{erfc}(aw)\operatorname{erfc}(bw)dw = \frac{1}{2}\int_0^t e^{-c^2/u}\frac{\operatorname{erfc}(a/\sqrt{u})\operatorname{erfc}(b/\sqrt{u})}{u^{3/2}}du$$ $$= J_3(a,b,c,T)$$ **(2.12)**

(1.49) $$I_3(a,b,c,T) = \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(cT) - I_5(c,a,T) - I_5(c,b,T) + J_3(a,b,c,T)$$ **(2.12)**

(1.50) $$\int_T^\infty w\ e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^2}\operatorname{erf}\left(a/\sqrt{u}\right)\operatorname{erf}\left(b/\sqrt{u}\right)du = J_4(a,b,c,T)$$ **(2.13)**

(1.51) $$\int_T^\infty w^2e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^{5/2}}\operatorname{erf}\left(\frac{a}{\sqrt{u}}\right)\operatorname{erf}\left(\frac{b}{\sqrt{u}}\right)du = I_4(a,b,c,T)$$ **(2.13)**

(1.52) $$\int_T^\infty e^{-c^2w^2}\frac{\operatorname{erf}(aw)\operatorname{erf}(bw)}{w^2}dw = \frac{1}{2}\int_0^t e^{-c^2/u}\frac{\operatorname{erf}(a/\sqrt{u})\operatorname{erf}(b/\sqrt{u})}{\sqrt{u}}dw = I_{14}(a,b,c,T)$$ **(2.14)**

Integrals Related To the Function

$$\mathbf{U(a,b,t) = e^{a^2t+2ab}erfc(a\sqrt{t} + b/\sqrt{t})}$$
$$\boldsymbol{a > 0,\quad b > 0,\quad c > 0,\quad d > 0,\quad t > 0}$$

(1.53) $$V(a,b,t) = \int_0^t U(a,b,\tau)d\tau,$$ **(2.15)**

(1.54) $$I_{21}(a,b,c,t) = \int_0^t U(a,b,\tau)\operatorname{erf}(c/\sqrt{\tau})d\tau$$ **(2.15)**

(1.55) $$I_{21}^c(a,b,c,t) = \int_0^t U(a,b,\tau)\operatorname{erfc}(c/\sqrt{\tau})d\tau$$ **(2.15)**

(1.56) $$J_{21}(a,b,c,t) = \int_0^t \frac{U(a,b,\tau)e^{-c^2/\tau}}{\tau^{3/2}}d\tau$$ **(2.15)**

(1.57) $$I_{22}(a,b,c,t) = \int_0^t U(a,b,\tau)\frac{e^{-c^2/\tau}}{\sqrt{\tau}}d\tau$$ **(2.16)**

(1.58) $$J_{22}(a,b,c,t) = \int_0^t U(a,b,\tau)\sqrt{\tau}e^{-c^2/\tau}\ d\tau$$ **(2.16)**

(1.59) $$I_{24}(a,b,c,t) = \int_0^t \tau U(a,b,\tau)\operatorname{erf}\left(\frac{c}{\sqrt{\tau}}\right)d\tau,$$ **(2.17)**

(1.60) $$I_{24}^c(a,b,c,t) = \int_0^t \tau U(a,b,\tau)\operatorname{erfc}\left(\frac{c}{\sqrt{\tau}}\right)d\tau,$$ **(2.17)**

(1.61) $$J_{24}(a,b,t) = \int_0^t \tau U(a,b,\tau)d\tau,$$ **(2.17)**

(1.62) $V_{24}(a,b,t) = \int_0^t V(a,b,\tau)d\tau$, **(2.17)**

(1.63) $I_{25}(a,b,c,d,t) = \int_0^t U(a,b,\tau)U(c,d,\tau)d\tau$ **(2.18)**

(1.64) $I_{26}(a,b,c,d,t) = \int_0^t \tau U(a,b,\tau)U(c,d,\tau)d\tau$ **(2.19)**

(1.65) $J = \int e^{(a^2-b^2)x}\mathrm{erfc}\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right)dx$ **(2.20)**

(1.66) $I = \int xe^{(a^2-b^2)x}\mathrm{erfc}\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right)dx$ **(2.20)**

Miscellaneous Integrals $x \geq 0$

(1.67) $H_{23}(x) = \int_0^x e^{w^2}\mathrm{erfc}(w)dw = \frac{1}{2}\int_0^{x^2} e^{v}\mathrm{erfc}(\sqrt{v})\frac{dv}{\sqrt{v}}$ **(2.21)**

(1.68) $I_{23}(x) = \int_0^x e^{w^2}\mathrm{erfc}(w)\,w\,dw = \frac{1}{2}\int_0^{x^2} e^{v}\mathrm{erfc}(\sqrt{v})dv$ **(2.21)**

(1.69) $J_{23}(x) = \int_0^x e^{w^2}\mathrm{erfc}(w)w^2dw = \frac{1}{2}\int_0^{x^2} e^{v}\mathrm{erfc}(\sqrt{v})\sqrt{v}dv$ **(2.21)**

Reduction formula ending in H_{23} or I_{23}

(1.70) $\int_0^x e^{w^2}\mathrm{erfc}(w)w^{\alpha}dw = \frac{x^{\alpha-1}}{2}e^{x^2}\mathrm{erfc}(x) + \frac{x^{\alpha}}{\alpha\sqrt{\pi}} - \frac{(\alpha-1)}{2}\int_0^x e^{w^2}\mathrm{erfc}(w)w^{\alpha-2}dw$ **(2.21)**

$\alpha = 2n$ or $2n+1$, n = 1, 2, …

(1.71) $G_n(a,b,x) = \int_x^{\infty} \frac{e^{-a^2w^2}i^n\mathrm{erfc}(bw)}{w^n}dw$, n = 1, 2, …, a>0, b>0 **(2.15)**

(1.72) $I(a,b,x) = \int_x^{\infty} e^{-at-b/t}dt$, $a > 0,\ b > 0,\ x > 0$ **(2.22)**

Inverse LaPlace transform:

(1.73) $L^{-1}\left[\frac{e^{-2b\sqrt{p}}}{\sqrt{p}(\sqrt{p}+a)^{n+1}}\right] = (2\sqrt{t})^n e^{a^2t+2ab}i^n\mathrm{erfc}(a\sqrt{t} + b/\sqrt{t})$

n=0 gives

(1.74) $L^{-1}\left[\frac{e^{-2b\sqrt{p}}}{\sqrt{p}(\sqrt{p}+a)}\right] = e^{a^2t+2ab}\mathrm{erfc}(a\sqrt{t} + b/\sqrt{t}) = U(a,b,t)$