

## General Solution Methods

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### Summary of Topics

- Separation of variables for linear & non-linear ODES.
- Homogeneous & non-homogeneous solutions of higher order linear ODES.
- Solving coupled ODEs.

### References

*Elementary Differential Equations & Boundary Value Problems*, 3<sup>rd</sup> ed.  
William Boyce & Richard DiPrima  
John Wiley & Sons, 1977

*Process Systems Analysis & Control*, 2<sup>nd</sup> ed.  
Donald R. Coughanowr  
McGraw-Hill, 1965

*Process Control — Designing Processes & Control Systems for Dynamic Performance*  
Thomas E. Marlin  
McGraw-Hill, 1995

We would like to find the analytical solution to the ODEs (ordinary differential equations) that result from the material & energy balances.

### Direct Integration of Separable Equations

Sometimes we can algebraically rearrange the ODE so that all of the expressions with one variable are on one side of the equation and all of the variables with the time dependency are on the other. In this case the ODE can be directly integrated. This separation of variables techniques can be applied equally to linear and non-linear ODEs. The real limitation is whether there are analytic forms for the resulting integrals.

For example, let's consider the liquid flow from a tank in which the flow through the outlet valve is dependent upon the liquid level in the tank:

$$A_1 \frac{dh_1}{dt} = F_0 - F_1 \quad \text{where } F_1 = F_1(h_1).$$

We would like an expression for  $h_1(t)$  for various  $F_0(t)$ .

### ***Linear Valve Characteristics***

If the valve has ***linear*** valve characteristics:

$$A_1 \frac{dh_1}{dt} = F_0 - C_v h_1 \quad \text{with the initial condition } F_0(0) = F_0^* \neq F_0(t > 0)$$

then the ODE can be rearranged as:

$$A_1 \frac{dh_1}{F_0 - C_v h_1} = dt.$$

If  $F_0$  is not a function of  $t$  (other than a change at  $t = 0$ ) then the variables have been separated and both sides can be integrated. For a step change  $F_0 \neq F_0^*$ :

$$\begin{aligned} A_1 \int_{h_1^*}^{h_1} \frac{dh_1}{F_0 - C_v h_1} &= \int_0^t dt \\ A_1 \left[ -\frac{1}{C_v} \ln(F_0 - C_v h_1) \right]_{h_1^*}^{h_1} &= t \\ \frac{A_1}{C_v} \ln \left( \frac{F_0 - C_v h_1}{F_0 - C_v h_1^*} \right) &= -t \\ \frac{F_0 - C_v h_1}{F_0 - C_v h_1^*} &= \exp \left( -\frac{C_v t}{A_1} \right) \\ h_1 &= \frac{F_0}{C_v} - \left( \frac{F_0}{C_v} - h_1^* \right) \exp \left( -\frac{C_v t}{A_1} \right) \\ &= \frac{F_0}{C_v} - \left( \frac{F_0 - F_0^*}{C_v} \right) \exp \left( -\frac{C_v t}{A_1} \right) \end{aligned}$$

Notice that one of the limits of integration is the initial level,  $h_1^*$ . And how do we determine this initial  $h_1^*$ ? Set the time derivative to zero for the steady state condition (remember, at steady state there is no accumulation) & insert the initial steady state values:

$$A_1 \frac{dh_1}{dt} = F_0 - C_v h_1 \Rightarrow 0 = F_0^* - C_v h_1^* \Rightarrow h_1^* = \frac{F_0^*}{C_v} \text{ at the initial steady state.}$$

**Non-Linear Valve Characteristics**

As a 2<sup>nd</sup> example, for the tank with **non-linear** valve characteristics:

$$A_1 \frac{dh_1}{dt} = F_0 - C_v \sqrt{h_1} \text{ with the initial condition } F_0(0) = F_0^* \neq F_0(t > 0)$$

Again, the initial  $h_1^*$  can be determined by setting the time derivative to zero & inserting the initial steady state values:

$$A_1 \frac{dh_1}{dt} = F_0 - C_v \sqrt{h_1} \Rightarrow 0 = F_0^* - C_v \sqrt{h_1^*} \Rightarrow h_1^* = \left( \frac{F_0^*}{C_v} \right)^2 \text{ at the initial steady state.}$$

Also again, if  $F_0$  is not a function of  $t$  (other than a change at  $t = 0$ ) then the variables can be separated and both sides can be integrated:

$$\begin{aligned} A_1 \frac{dh_1}{F_0 - C_v \sqrt{h_1}} &= dt \\ A_1 \int_{h_1^*}^{h_1} \frac{dh_1}{F_0 - C_v \sqrt{h_1}} &= \int_0^t dt \\ A_1 \left[ -\frac{2\sqrt{h_1}}{C_v} - \frac{2F_0}{C_v^2} \ln(F_0 - C_v \sqrt{h_1}) \right]_{h_1^*}^{h_1} &= t \\ A_1 \left[ -\frac{2(\sqrt{h_1} - \sqrt{h_1^*})}{C_v} - \frac{2F_0}{C_v^2} \ln \left( \frac{F_0 - C_v \sqrt{h_1}}{F_0 - C_v \sqrt{h_1^*}} \right) \right] &= t \\ (\sqrt{h_1} - \sqrt{h_1^*}) + \frac{F_0}{C_v} \ln \left( \frac{F_0 - C_v \sqrt{h_1}}{F_0 - C_v \sqrt{h_1^*}} \right) &= -\frac{C_v t}{2A_1} \\ \left( \sqrt{h_1} - \frac{F_0^*}{C_v} \right) + \frac{F_0}{C_v} \ln \left( \frac{F_0 - C_v \sqrt{h_1}}{F_0 - F_0^*} \right) &= -\frac{C_v t}{2A_1} \end{aligned}$$

Notice that we have not found an **explicit** solution  $h_1(t)$  but rather an **implicit** one. When plotting, it is easier to calculate  $t(h_1)$  for the range  $h_1^* \leq h_1 \leq (F_0/C_v)^2$ .

## Integration Factor

The integrating factor can be used to integrate a **1<sup>st</sup> order linear ODE** where the coefficients can be functions of  $t$ . Consider:

$$a(t)\frac{dY}{dt} + b(t)Y = c(t)$$

When  $a(t) \neq 0$  over the range of interest, then the equation can be rearranged to give:

$$\frac{dY}{dt} + f(t)Y = g(t)$$

where  $g(t)$  is the forcing function. The ODE is linear but not separable. However, we can make it separable by defining an **integrating factor**:

$$\mathcal{F} \equiv \exp\left(\int f(t)dt\right)$$

If we multiply the ODE by the integrating factor:

$$\exp\left(\int f(t)dt\right)\frac{dY}{dt} + f(t)\exp\left(\int f(t)dt\right)Y = g(t)\exp\left(\int f(t)dt\right)$$

The left-hand side of this equation can be recognized as the expansion of the derivative of the product:

$$\frac{d}{dt}\left[\exp\left(\int f(t)dt\right)Y\right] = \exp\left(\int f(t)dt\right)\frac{dY}{dt} + f(t)\exp\left(\int f(t)dt\right)Y$$

So, the ODE is now:

$$\frac{d}{dt}\left[\exp\left(\int f(t)dt\right)Y\right] = g(t)\exp\left(\int f(t)dt\right)$$

and can be solved to give:

$$\begin{aligned}\int d\left[\exp\left(\int f(t)dt\right)Y\right] &= \int g(t)\exp\left(\int f(t)dt\right)dt \\ \exp\left(\int f(t)dt\right)Y &= \int g(t)\exp\left(\int f(t)dt\right)dt + I_o \\ Y &= \exp\left(-\int f(t)dt\right)\int g(t)\exp\left(\int f(t)dt\right)dt + I_o \exp\left(-\int f(t)dt\right)\end{aligned}$$

where  $I_o$  is a constant of integration to be determined from the initial conditions. Let's apply this to the tank flow with linear valve characteristics problem:

$$A_1 \frac{dh_1}{dt} = F_0 - C_v h_1 \quad \text{with the initial condition } F_0(0) = F_0^* \neq F_0(t > 0)$$

$$\frac{dh_1}{dt} + \frac{C_v}{A_1} h_1 = \frac{F_0}{A_1}$$

here:  $f(t) = \frac{C_v}{A_1} \Rightarrow \exp\left(\int f(t) dt\right) = \exp\left(\frac{C_v t}{A_1}\right)$

$$g(t) = \frac{F_0}{A_1}$$

and:

$$h_1 = \exp\left(-\frac{C_v t}{A_1}\right) \int_0^t \frac{F_0}{A_1} \exp\left(\frac{C_v t}{A_1}\right) dt + h_1^* \exp\left(-\frac{C_v t}{A_1}\right)$$

$$h_1 = \exp\left(-\frac{C_v t}{A_1}\right) \frac{F_0}{C_v} \left[ \exp\left(\frac{C_v t}{A_1}\right) - 1 \right] + h_1^* \exp\left(-\frac{C_v t}{A_1}\right)$$

$$h_1 = \frac{F_0}{C_v} \left[ 1 - \exp\left(-\frac{C_v t}{A_1}\right) \right] + h_1^* \exp\left(-\frac{C_v t}{A_1}\right)$$

$$h_1 = \frac{F_0}{C_v} - \left( \frac{F_0}{C_v} - h_1^* \right) \exp\left(-\frac{C_v t}{A_1}\right)$$

This is the same result as found previously.

## Higher Order Linear ODEs

When the coefficients are constant, then the solution can be determined in two parts: the general homogeneous solution and the specific non-homogeneous part. Let's assume we have the homogeneous problem:

$$a \frac{d^2 Y}{dt^2} + b \frac{dY}{dt} + cY = 0$$

Let's assume a solution of the form  $Y(t) = e^{rt}$ . Then:

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

$$e^{rt} [ar^2 + br + c] = 0$$

Since  $e^{rt} \neq 0$  then:

$$ar^2 + br + c = 0$$

There will actually be two roots (which we can get using the quadratic equation rule):

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The total solution will be a linear combination of these two particular solutions:

$$Y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where the constants are determined from the boundary conditions.

Note that the form of the solution will be different if there are repeated roots. (This will not be discussed here, though.)

If we have the general non-homogeneous problem:

$$a \frac{d^2 Y}{dt^2} + b \frac{dY}{dt} + cY = f(t)$$

then the solution process is more complex. There are two steps:

1. Find the solution to the homogeneous problem,  $Y_c(t)$ .
2. Find a particular solution to the full problem,  $Y_p(t)$ . This can be done with ***the method of undetermined coefficients***.
3. The general solution to the full problem will be the sum of these two,  $Y(t) = Y_c(t) + Y_p(t)$ .

For example, let's find the solution to:

$$\frac{d^2 Y}{dt^2} - 3 \frac{dY}{dt} - 4Y = 4t^2$$

(1) The homogeneous solution comes from the roots of:

$$r^2 - 3r - 4 = 0 \Rightarrow r_1, r_2 = 4, -1 \Rightarrow Y_c(t) = C_1 e^{4t} + C_2 e^{-t}.$$

(2) Next, let's assume a particular solution of the form  $Y_p(t) = At^2 + Bt + C$ . (It should have the highest order of the polynomial present & all lower order terms.) Then:

$$(2A) - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2$$
$$(2A - 3B - 4C) - (6A + 4B)t - (4A)t^2 = 4t^2$$

Comparing terms gives us three equations:

$$\begin{aligned} -4A &= 4 \\ 6A + 4B &= 0 \\ 2A - 3B - 4C &= 0 \end{aligned}$$

We have the solution  $A = -1$ ,  $B = 3/2$ , and  $C = -13/8$ . This gives the particular solution:

$$Y_p(t) = -t^2 + \frac{3}{2}t - \frac{13}{8}$$

(3) Finally we add these two together to get the total solution:

$$Y(t) = C_1 e^{4t} + C_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{13}{8}.$$

There are various rules for picking the form of the particular solution when  $f(t)$  has the following forms:

$$f(t) = \begin{cases} P_n(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_n \\ e^{\alpha t} P_n(t) \\ e^{\alpha t} P_n(t) \sin \beta t \\ e^{\alpha t} P_n(t) \cos \beta t \end{cases}$$

but we will not discuss these here.

### **Solution of Coupled ODEs**

We have seen cases where there are two or more variables that must be solved simultaneously since the variables are in each equation. The goal is to do some type of algebraic and/or calculus manipulation so that an equation can be created that only depends upon one variable.

Let's assume we have the ODEs:

$$\frac{dY_1}{dt} = 10 - 7Y_1 + Y_2$$
$$\frac{dY_2}{dt} = -6Y_2 + 2Y_1$$

Solve the 2<sup>nd</sup> ODE for  $Y_1$  :

$$Y_1 = \frac{1}{2} \frac{dY_2}{dt} + 3Y_2$$

and differentiate once:

$$\frac{dY_1}{dt} = \frac{1}{2} \frac{d^2Y_2}{dt^2} + 3 \frac{dY_2}{dt}$$

Insert these expressions (the original & the differentiated form) into the 1<sup>st</sup> ODE:

$$\left[ \frac{1}{2} \frac{d^2Y_2}{dt^2} + 3 \frac{dY_2}{dt} \right] = 10 - 7 \left[ \frac{1}{2} \frac{dY_2}{dt} + 3Y_2 \right] + Y_2$$
$$\frac{d^2Y_2}{dt^2} + 13 \frac{dY_2}{dt} + 40Y_2 = 20$$

This 2<sup>nd</sup> order ODE can be solved:

$$Y_2 = C_1 e^{-8t} + C_2 e^{-5t} + \frac{1}{2}$$

$Y_1$  can then be obtained from the manipulated form of the original 2<sup>nd</sup> ODE:

$$Y_1 = \frac{1}{2} \frac{dY_2}{dt} + 3Y_2 = \frac{1}{2} (-8C_1 e^{-8t} - 5C_2 e^{-5t}) + 3 \left( C_1 e^{-8t} + C_2 e^{-5t} + \frac{1}{2} \right)$$
$$= -C_1 e^{-8t} + \frac{1}{2} C_2 e^{-5t} + \frac{3}{2}$$