Useful Laplace Transforms & Hyperbolic/Trigonometric Functions

Definition:

$$\mathcal{L}\left[f(t)\right] \equiv \overline{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$$

Linearity:

$$\mathcal{L}\left[a\cdot f_1(t)+b\cdot f_2(t)\right] = a\cdot\mathcal{L}\left[f_1(t)\right]+b\cdot\mathcal{L}\left[f_2(t)\right] = a\cdot\overline{f}_1(s)+b\cdot\overline{f}_2(s)$$

Transform of derivatives:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = s\cdot\overline{f}(s)-f(0)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2\cdot\overline{f}(s)-s\cdot f(0)-f'(0)$$

$$\mathcal{L}\left[\frac{d^{(n)}f(t)}{dt^{(n)}}\right] = s^n\cdot\overline{f}(s)-s^{n-1}\cdot f(0)-s^{n-2}\cdot f'(0)-\cdots-sf^{(n-2)}\cdot(0)-f^{(n-1)}(0)$$

Transform of integral:

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\overline{f}(s)$$

Limits:

$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[s\cdot\overline{f}(s)\right]$	
$\lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}\left[s\cdot\overline{f}(s)\right]$	Valid only if $s\cdot\overline{f}(s)$ does not become infinite for any value of s satisfying $\mathrm{Re}(s)\ge 0$.

Translation of transform:

$$\mathcal{L}\left[e^{-at}f(t)\right] = \overline{f}(s+a) \quad\Rightarrow\quad \mathcal{L}^{-1}\left[\overline{f}(s+a)\right] = e^{-at}f(t)$$

Translation of function:

$$\mathcal{L}\left[f(t-t_0)\cdot\mathrm{u}(t-t_0)\right] = e^{-st_0}\overline{f}(s) \quad\Rightarrow\quad \mathcal{L}^{-1}\left[e^{-st_0}\overline{f}(s)\right] = f(t-t_0)\cdot\mathrm{u}(t-t_0)$$

Exponential forms of hyperbolic & trigonometric (Euler) functions:

$\sinh(x) = \frac{1}{2}\left(e^x-e^{-x}\right)$	$\sin(x) = -\frac{j}{2}\left(e^{jx}-e^{-jx}\right)$
$\cosh(x) = \frac{1}{2}\left(e^x+e^{-x}\right)$	$\cos(x) = \frac{1}{2}\left(e^{jx}+e^{-jx}\right)$

Trigonometric Identity:

$$p\cos\theta+q\sin\theta = r\sin(\theta+\phi) \text{ where } r=\sqrt{p^2+q^2} \text{ and } \tan\phi = \frac{p}{q}$$

Laplace Transform Pairs

$f(t)$	$\bar{f}(s)$
$e^{-at} f(t)$	$\bar{f}(s+a)$
$f(t-t_0)\cdot \mathrm{u}(t-t_0)$	$e^{-st_0}\bar{f}(s)$
$t \cdot f(t)$	$-\dfrac{d\bar{f}(s)}{ds}$
1	$\dfrac{1}{s}$
t^n, $n=0,1,2,\ldots$	$\dfrac{n!}{s^{n+1}}$
t^ν, $\nu>1$	$\dfrac{\Gamma(\nu+1)}{s^{\nu+1}}$
$t^{n-1/2}$	$\left(\dfrac{\sqrt{\pi}}{2}\right)\left(\dfrac{3}{2}\right)\left(\dfrac{5}{2}\right)\cdots\left(\dfrac{n-1}{2}\right)\dfrac{1}{s^{n+1/2}}$
e^{-at}	$\dfrac{1}{s+a}$
e^{+at}	$\dfrac{1}{s-a}$
$\dfrac{t^{n-1}e^{-at}}{(n-1)!}$, $n=1,2,3,\ldots$	$\dfrac{1}{(s+a)^n}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
$\mathrm{u}(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$	$\dfrac{e^{-as}}{s}$, $a\geq 0$
$\delta(t-a)$	e^{-as}, $a\geq 0$