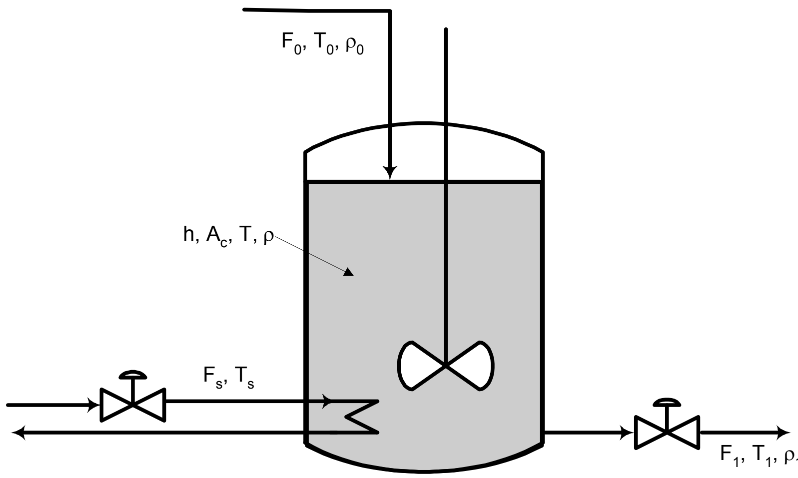
Developing Transfer Functions from Heat & Material Balances

Example Transfer Functions — Stirred Tank Heater



We will develop the transfer functions for a stirred tank heater by setting up the heat & material balance equations for the case where the volume might change (since flow out of the tank is controlled by a valve). The overall mass balance on this system will be:

$$\frac{d(\rho_1 V)}{dt} = \rho_0 F_0 - \rho_1 F_1$$

where the volume in the tank, V, might be changing with time (since the liquid level might be changing). If we simplify the physical properties by assuming a constant liquid density, $\rho_1 = \rho_0 \equiv \rho$, and if the valve has linear flow characteristics, $F_1 = C_v h$, then:

$$\frac{d(\rho A_c h)}{dt} = \rho F_0 - \rho F_1 \quad \Rightarrow \quad A_c \frac{dh}{dt} = F_0 - C_v h .$$

Remember that the steady state version of this equation is:

$$0 = F_0^* - C_v h^* .$$

The energy balance will be:

$$\frac{d\left(\rho_1 V \hat{H}_1\right)}{dt} = \rho_0 F_0 \hat{H}_0 - \rho_1 F_1 \hat{H}_1 + UA\left(T_s - T_1\right)$$

We can split apart the time derivative term:

$$\rho_1 V \frac{d\left(\hat{H}_1\right)}{dt} + \hat{H}_1 \frac{d\left(\rho_1 V\right)}{dt} = \rho_0 F_0 \hat{H}_0 - \rho_1 F_1 \hat{H}_1 + UA\left(T_s - T_1\right)$$

and then insert the mass balance ODE for the second time derivative:

$$\rho_1 V \frac{d\left(\hat{H}_1\right)}{dt} + \hat{H}_1 \left[F_0 \rho_0 - F_1 \rho_1\right] = \rho_0 F_0 \hat{H}_0 - \rho_1 F_1 \hat{H}_1 + UA\left(T_s - T_1\right).$$

Doing a little bit of math gives:

$$\rho_1 V \frac{d\left(\hat{H}_1\right)}{dt} + F_0 \rho_0 \hat{H}_1 - F_1 \rho_1 \hat{H}_1 = \rho_0 F_0 \hat{H}_0 - \rho_1 F_1 \hat{H}_1 + UA\left(T_s - T_1\right)$$

$$\rho_1 V \frac{d\left(\hat{H}_1\right)}{dt} + F_0 \rho_0 \hat{H}_1 = \rho_0 F_0 \hat{H}_0 + UA\left(T_s - T_1\right)$$

$$\rho_1 V \frac{d\left(\hat{H}_1\right)}{dt} = \rho_0 F_0 \hat{H}_0 - F_0 \rho_0 \hat{H}_1 + UA\left(T_s - T_1\right)$$

$$\rho_1 V \frac{d\hat{H}_1}{dt} = \rho_0 F_0 \left(\hat{H}_0 - \hat{H}_1\right) + UA\left(T_s - T_1\right).$$

We would still like to convert the enthalpy terms into terms that directly relate to temperature. Even with a temperature dependent heat capacity the time derivative can be split into:

$$\rho_1 V \hat{C}_{p1} \frac{dT_1}{dt} = \rho_0 F_0 \left(\hat{H}_0 - \hat{H}_1\right) + UA\left(T_s - T_1\right).$$

If we assume that the heat capacities are constant and that there is no reaction, then:

$$\hat{H}\left(T\right) = \hat{C}_p \left(T - T_{ref}\right) + \hat{H}_{ref} \quad \text{and} \quad \hat{H}_{ref,1} = \hat{H}_{ref,0}$$

and:

$$\rho_1 V \hat{C}_p \frac{dT_1}{dt} = \rho_0 F_0 \hat{C}_p \left(T_0 - T_1\right) + UA\left(T_s - T_1\right).$$

For final simplicity let's assume the density is essentially constant, so $\rho_1 = \rho_0 \equiv \rho$ and:

$$\rho V\hat{C}_p \frac{dT_1}{dt} = \rho F_0 \hat{C}_p \left(T_0 - T_1\right) + UA\left(T_s - T_1\right).$$

The steady state version of this equation is:

$$0 = \rho F_0^* \hat{C}_p \left(T_0^* - T_1^*\right) + UA\left(T_s^* - T_1^*\right).$$

Changing temperatures, constant inlet flow rate

Let's develop the transfer functions using these ODEs as our starting point. Let's assume that the only independent variables that can change are T_0 and T_s. We will hold F_0 constant, so V (and hence h) will be constant, too. So we only need to deal with the energy balance equation. When we linearize this we need to put it in terms of the deviation variables T_0' and T_s' (for the independent variables) and T_1' (for the dependent variable):

$$\begin{aligned}\left[\rho V\hat{C}_p\right]\frac{dT_1'}{dt} &= \frac{\partial}{\partial T_0}\left[\rho F_0 \hat{C}_p \left(T_0 - T_1\right) + UA\left(T_s - T_1\right)\right]^* \cdot T_0' \\ &+ \frac{\partial}{\partial T_s}\left[\rho F_0 \hat{C}_p \left(T_0 - T_1\right) + UA\left(T_s - T_1\right)\right]^* \cdot T_s' \\ &+ \frac{\partial}{\partial T_1}\left[\rho F_0 \hat{C}_p \left(T_0 - T_1\right) + UA\left(T_s - T_1\right)\right]^* \cdot T_1'\end{aligned}$$

$$\left[\rho V\hat{C}_p\right]\frac{dT_1'}{dt} = \left[\rho F_0 \hat{C}_p\right]^* \cdot T_0' + \left[UA\right]^* \cdot T_s' + \left[-\rho F_0 \hat{C}_p - UA\right]^* \cdot T_1'$$

$$\left[\rho V\hat{C}_p\right]\frac{dT_1'}{dt} + \left[\rho F_0 \hat{C}_p + UA\right] \cdot T_1' = \left[\rho F_0 \hat{C}_p\right] \cdot T_0' + \left[UA\right] \cdot T_s'.$$

Note that this equation shows how the stirred tank fluid temperature is affected by <u>changes</u> in the other temperatures.

This ODE gets converted to transfer functions by taking the Laplace transform :

$$\left[\rho V\hat{C}_p\right] s \cdot \overline{T}_1' + \left[\rho F_0 \hat{C}_p + UA\right] \cdot \overline{T}_1' = \left[\rho F_0 \hat{C}_p\right] \cdot \overline{T}_0' + \left[UA\right] \cdot \overline{T}_s'$$

$$\left(\left[\rho V\hat{C}_p\right] s + \left[\rho F_0 \hat{C}_p + UA\right]\right) \cdot \overline{T}_1' = \left[\rho F_0 \hat{C}_p\right] \cdot \overline{T}_0' + \left[UA\right] \cdot \overline{T}_s'$$

$$\overline{T}_1' = \frac{\left[\rho F_0 \hat{C}_p\right]}{\left[\rho V\hat{C}_p\right] s + \left[\rho F_0 \hat{C}_p + UA\right]} \cdot \overline{T}_0' + \frac{\left[UA\right]}{\left[\rho V\hat{C}_p\right] s + \left[\rho F_0 \hat{C}_p + UA\right]} \cdot \overline{T}_s'.$$

This shows that we have two transfer functions:

$$\overline{T}_1' = G_0(s)\overline{T}_0' + G_s(s)\overline{T}_s'$$

where:

$$G_0(s) \equiv \frac{\left[\rho F_0 \hat{C}_p\right]}{\left[\rho V \hat{C}_p\right]s + \left[\rho F_0 \hat{C}_p + UA\right]} \text{ and } G_s(s) \equiv \frac{[UA]}{\left[\rho V \hat{C}_p\right]s + \left[\rho F_0 \hat{C}_p + UA\right]}.$$

Notice that both of these transfer functions come from the same first order ODE, so both are referred to as first order transfer functions. We'll see in a later section that we would like to put the equations into a standard form that arises when the multiplier on the $\overline{T}_1'$ term is unity (i.e., one). This can be done by dividing everything by the existing factor on the $\overline{T}_1'$ term, $\left[\rho F_0 \hat{C}_p + UA\right]$. Starting with the deviation variable form of the ODE:

$$\left[\frac{\rho V \hat{C}_p}{\rho F_0 \hat{C}_p + UA}\right]\frac{dT_1'}{dt} + T_1' = \left[\frac{\rho F_0 \hat{C}_p}{\rho F_0 \hat{C}_p + UA}\right]\cdot T_0' + \left[\frac{UA}{\rho F_0 \hat{C}_p + UA}\right]\cdot T_s'.$$

We can simplify the form of the equation by defining new symbols for the groupings:

$$\tau \equiv \frac{\rho V \hat{C}_p}{\rho F_0 \hat{C}_p + UA}., \; K_0 \equiv \frac{\rho F_0 \hat{C}_p}{\rho F_0 \hat{C}_p + UA}, \text{ and } K_s \equiv \frac{UA}{\rho F_0 \hat{C}_p + UA}$$

so:

$$\tau \cdot \frac{dT_1'}{dt} + T_1' = K_0 \cdot T_0' + K_s \cdot T_s'$$

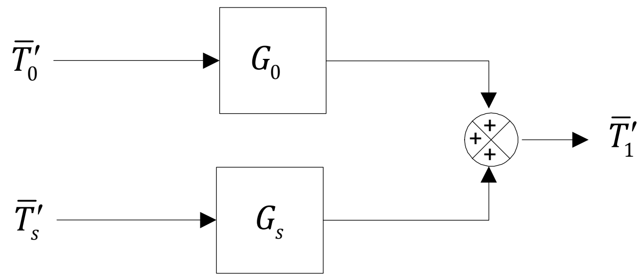
$$(\tau s + 1)\overline{T}_1' = K_0 \cdot \overline{T}_0' + K_s \cdot \overline{T}_s'$$

$$\overline{T}_1' = \frac{K_0}{\tau s + 1}\cdot \overline{T}_0' + \frac{K_s}{\tau s + 1}\cdot \overline{T}_s'$$

and the transfer functions are now:

$$G_0(s) = \frac{K_0}{\tau s + 1} \text{ and } G_s(s) = \frac{K_s}{\tau s + 1}.$$

A block diagram for the stirred tank heater can be drawn as follows.



Allow Inlet Flow Rate to Change

The previous set of transfer functions were derived from the energy balance ODE linearized only for changes in temperature. If the inlet flow rate also changes then we must also take this into account in both ODEs. When we linearize these ODEs we need to put them in terms of the deviation variables F_0', T_0', and T_s' (for the independent variables) and h' & T_1' (for the dependent variables). The overall material balance becomes:

$$\begin{aligned} A_c \frac{dh'}{dt} &= \frac{\partial}{\partial F_0}\left[F_0 - C_v h\right]^* \cdot F_0' \\ &+ \frac{\partial}{\partial T_0}\left[F_0 - C_v h\right]^* \cdot T_0' \\ &+ \frac{\partial}{\partial T_s}\left[F_0 - C_v h\right]^* \cdot T_s' \\ &+ \frac{\partial}{\partial h}\left[F_0 - C_v h\right]^* \cdot h' \\ &+ \frac{\partial}{\partial T}\left[F_0 - C_v h\right]^* \cdot T' \end{aligned}$$

$$A_c \frac{dh'}{dt} = [1] \cdot F_0' + [0] \cdot T_0' + [0] \cdot T_s' + [-C_v] \cdot h' + [0] \cdot T'$$

$$A_c \frac{dh'}{dt} = F_0' - C_v h'$$

and: $\quad [A_c] \dfrac{dh'}{dt} + [C_v] h' = F_0'$.

The energy balance becomes (remember to first divide by h to put all variables on the left-hand side of the ODE):

$$\left[\rho A_c \hat{C}_p\right]\frac{dT_1}{dt} = \frac{\partial}{\partial F_0}\left[\frac{\rho F_0 \hat{C}_p}{h}\left(T_0 - T_1\right) + \frac{UA}{h}\left(T_s - T_1\right)\right]^* \cdot F_0'$$

$$+\frac{\partial}{\partial T_0}\left[\frac{\rho F_0 \hat{C}_p}{h}\left(T_0 - T_1\right) + \frac{UA}{h}\left(T_s - T_1\right)\right]^* \cdot T_0'$$

$$+\frac{\partial}{\partial T_s}\left[\frac{\rho F_0 \hat{C}_p}{h}\left(T_0 - T_1\right) + \frac{UA}{h}\left(T_s - T_1\right)\right]^* \cdot T_s'$$

$$+\frac{\partial}{\partial h}\left[\frac{\rho F_0 \hat{C}_p}{h}\left(T_0 - T_1\right) + \frac{UA}{h}\left(T_s - T_1\right)\right]^* \cdot h'$$

$$+\frac{\partial}{\partial T_1}\left[\frac{\rho F_0 \hat{C}_p}{h}\left(T_0 - T_1\right) + \frac{UA}{h}\left(T_s - T_1\right)\right]^* \cdot T_1'$$

$$\left[\rho A_c \hat{C}_p\right]\frac{dT_1}{dt} = \left[\frac{\rho \hat{C}_p}{h}\left(T_0 - T_1\right)\right]^* \cdot F_0' + \left[\frac{\rho F_0 \hat{C}_p}{h}\right]^* \cdot T_0' + \left[\frac{UA}{h}\right]^* \cdot T_s'$$

$$+\left[-\frac{\rho F_0 \hat{C}_p}{h^2}\left(T_0 - T_1\right) - \frac{UA}{h^2}\left(T_s - T_1\right)\right]^* \cdot h'$$

$$+\left[-\frac{\rho F_0 \hat{C}_p}{h} - \frac{UA}{h}\right]^* \cdot T_1'$$

$$\left[\rho A_c \hat{C}_p\right]\frac{dT_1}{dt} = \left[\frac{\rho \hat{C}_p}{h^*}\left(T_0^* - T_1^*\right)\right] \cdot F_0' + \left[\frac{\rho F_0^* \hat{C}_p}{h^*}\right] \cdot T_0' + \left[\frac{UA}{h^*}\right] \cdot T_s'$$

$$+\left[-\frac{\rho F_0^* \hat{C}_p}{h^{*2}}\left(T_0^* - T_1^*\right) - \frac{UA}{h^{*2}}\left(T_s^* - T_1^*\right)\right] \cdot h'$$

$$+\left[-\frac{\rho F_0^* \hat{C}_p}{h^*} - \frac{UA}{h^*}\right] \cdot T_1'$$

$$\left[\rho A_c \hat{C}_p\right]\frac{dT_1}{dt} + \left[\frac{\rho F_0^* \hat{C}_p}{h^*} + \frac{UA}{h^*}\right] \cdot T_1' = \left[\frac{\rho \hat{C}_p}{h^*}\left(T_0^* - T_1^*\right)\right] \cdot F_0' + \left[\frac{\rho F_0^* \hat{C}_p}{h^*}\right] \cdot T_0' + \left[\frac{UA}{h^*}\right] \cdot T_s'$$

$$-\left[\frac{\rho F_0^* \hat{C}_p}{h^{*2}}\left(T_0^* - T_1^*\right) + \frac{UA}{h^{*2}}\left(T_s^* - T_1^*\right)\right] \cdot h'$$

We can multiply through by the steady state level, h^*:

$$\left[\rho A_c h^* \hat{C}_p\right]\frac{dT_1}{dt}+\left[\rho F_0^* \hat{C}_p + UA\right]\cdot T_1' = \left[\rho \hat{C}_p\left(T_0^* - T_1^*\right)\right]\cdot F_0' + \left[\rho F_0^* \hat{C}_p\right]\cdot T_0' + \left[UA\right]\cdot T_s' - \left[\frac{\rho F_0^* \hat{C}_p\left(T_0^* - T_1^*\right) + UA\left(T_s^* - T_1^*\right)}{h^*}\right]\cdot h'$$

Notice that <u>the term multiplying h' is the steady state energy balance and is zero</u>, so:

$$\left[\rho A_c h^* \hat{C}_p\right]\frac{dT_1}{dt}+\left[\rho F_0^* \hat{C}_p + UA\right]\cdot T_1' = \left[\rho \hat{C}_p\left(T_0^* - T_1^*\right)\right]\cdot F_0' + \left[\rho F_0^* \hat{C}_p\right]\cdot T_0' + \left[UA\right]\cdot T_s'.$$

Now we can get the transfer functions by converting to Laplace variables & algebraically manipulating. First from the mass balance:

$$\left[A_c\right]\cdot s\cdot \overline{h}' + \left[C_v\right]\cdot \overline{h}' = \overline{F}_0'$$

$$\left(A_c s + C_v\right)\cdot \overline{h}' = \overline{F}_0'$$

$$\overline{h}' = \frac{1}{A_c s + C_v}\overline{F}_0'$$

Next from the energy balance:

$$\left(\left[\rho A_c h^* \hat{C}_p\right]s + \left[\rho F_0^* \hat{C}_p + UA\right]\right)\cdot \overline{T}_1' = \left[\rho \hat{C}_p\left(T_0^* - T_1^*\right)\right]\cdot \overline{F}_0' + \left[\rho F_0^* \hat{C}_p\right]\cdot \overline{T}_0' + \left[UA\right]\cdot \overline{T}_s'$$

$$\overline{T}_1' = \frac{\left[\rho \hat{C}_p\left(T_0^* - T_1^*\right)\right]}{\left[\rho A_c h^* \hat{C}_p\right]s + \left[\rho F_0^* \hat{C}_p + UA\right]}\cdot \overline{F}_0' + \frac{\left[\rho F_0^* \hat{C}_p\right]}{\left[\rho A_c h^* \hat{C}_p\right]s + \left[\rho F_0^* \hat{C}_p + UA\right]}\cdot \overline{T}_0' + \frac{\left[UA\right]}{\left[\rho A_c h^* \hat{C}_p\right]s + \left[\rho F_0^* \hat{C}_p + UA\right]}\cdot \overline{T}_s'$$

Now there are three transfer functions from the energy balance:

$$\overline{T}_1' = G_F \cdot \overline{F}_0' + G_0 \cdot \overline{T}_0' + G_s \cdot \overline{T}_s'$$

where:

$$G_F \equiv \frac{\left[\rho \hat{C}_p \left(T_0^* - T_1^*\right)\right]}{\left[\rho A_c h^* \hat{C}_p\right] s + \left[\rho F_0^* \hat{C}_p + UA\right]}$$

$$G_0(s) = \frac{\left[\rho F_0^* \hat{C}_p\right]}{\left[\rho A_c h^* \hat{C}_p\right] s + \left[\rho F_0^* \hat{C}_p + UA\right]}$$

$$G_s(s) = \frac{[UA]}{\left[\rho A_c h^* \hat{C}_p\right] s + \left[\rho F_0^* \hat{C}_p + UA\right]}.$$

There is only one transfer function from the mass balance:

$$\overline{h}' = G_h \cdot \overline{F}_0' \quad \text{where } G_h \equiv \frac{1}{A_c s + C_v}.$$

The full block diagram relating the independent & dependent variables are shown in the following figure.