Linear Open Loop Systems

1st Order Systems

Output modeled with a 1st order ODE:

$$a_1 \frac{dy}{dt} + a_0 y = b \cdot f(t).$$

If $a_0 \neq 0$, then:

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0} f(t) \quad \Rightarrow \quad \tau_p \frac{dy}{dt} + y = K_p \cdot f(t)$$

where: τ_p is the *time constant*.
K_p is the *steady state gain*, *static gain*, or *gain*.

For deviation variables, where $y(0) = f(0) = 0$, the Laplace transform will be:

$$(\tau_p s + 1)\bar{y}(s) = K_p \bar{f}(s) \quad \Rightarrow \quad G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1}$$

This transfer function is referred to as *1st order lag*, *linear lag*, or *exponential transfer lag*.

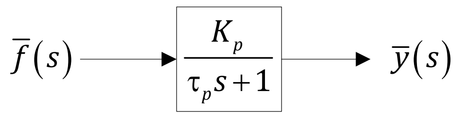
But what if $a_0 = 0$? Then:

$$a_1 \frac{dy}{dt} = bf(t) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{b}{a_1} f(t) \quad \Rightarrow \quad \frac{dy}{dt} = K'_p \cdot f(t).$$

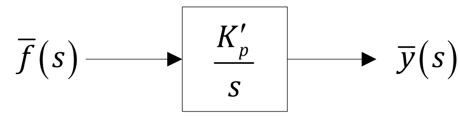
For deviation variables, the Laplace transform will be:

$$s\bar{y}(s) = K_p' \cdot \bar{f}(s) \quad \Rightarrow \quad G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p'}{s}.$$

This transfer function is referred to as *purely capacitive* or *pure integrator*.



1st Order lag



Pure Integrator

Example 1st Order Systems — Mercury Thermometer

Last time we developed the following equation for the reading from a mercury thermometer:

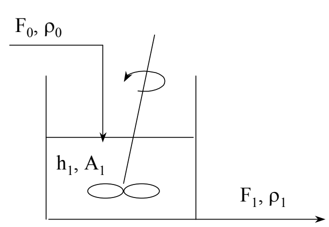
$$\frac{m\hat{C}_p}{hA}\frac{dT}{dt} = T_a - T \quad \Rightarrow \quad \frac{m\hat{C}_p}{hA}\frac{dT}{dt} + T = T_a$$

So this is a 1st order lag system with:

$$\tau_p = \frac{m\hat{C}_p}{hA}$$
$$K_p = 1$$

Example 1st Order Systems — Mass Storage in Tank

Mass storage in a tank is a 1st order system, but we don't know which type until we say something about how the flow out of the tank is controlled.



For constant density & constant cross-sectional area:

$$\frac{d(\rho_1 A h)}{dt} = \rho_0 F_0 - \rho_1 F_1 \quad \Rightarrow \quad A\frac{dh}{dt} = F_0 - F_1.$$

For flow through a valve where we can linearize to $F_1 \approx C_v h = h/R$, then:

$$A\frac{dh}{dt} = F_0 - \frac{h}{R} \quad \Rightarrow \quad AR\frac{dh}{dt} + h = RF_0.$$

So this is a 1st order lag system with:

$$\tau_p = AR$$
$$K_p = R$$

However, if the flowrate out is controlled separately from the level in the tank, e.g., with a pump, then:

$$A\frac{dh}{dt} = F_0 - F_1 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{F_0 - F_1}{A}.$$

So this is <u>pure integrator</u> system with:

$$f(t) = F_0 - F_1$$
$$K_p' = \frac{1}{A}.$$

Response of 1st Order Systems

Look at response to 4 typical driving functions.

Impulse disturbance. $f(t) = \alpha \cdot \delta(0) \quad \Rightarrow \quad \overline{f}(s) = \alpha$. So, if 1st order lag:

$$\overline{y}(s) = G(s)\cdot\overline{f}(s) = \frac{K_p}{\tau_p s + 1}\alpha = \frac{\dfrac{\alpha K_p}{\tau_p}}{s + \dfrac{1}{\tau_p}} \quad \Rightarrow \quad y(t) = \alpha\frac{K_p}{\tau_p}e^{-t/\tau_p}$$

If pure integrator:

$$\overline{y}(s) = G(s)\cdot\overline{f}(s) = \frac{K_p'}{s}\alpha \quad \Rightarrow \quad y(t) = \alpha K_p'$$

Unit step change. $f(t) = \alpha \cdot H(t) \quad \Rightarrow \quad \overline{f}(s) = \alpha/s$. So, if 1st order lag:

$$\bar{y}(s) = G(s)\cdot\bar{f}(s) = \frac{K_p}{\tau_p s+1}\frac{\alpha}{s} = \alpha K_p\left(\frac{1}{s} - \frac{\tau_p}{\tau_p s+1}\right)$$

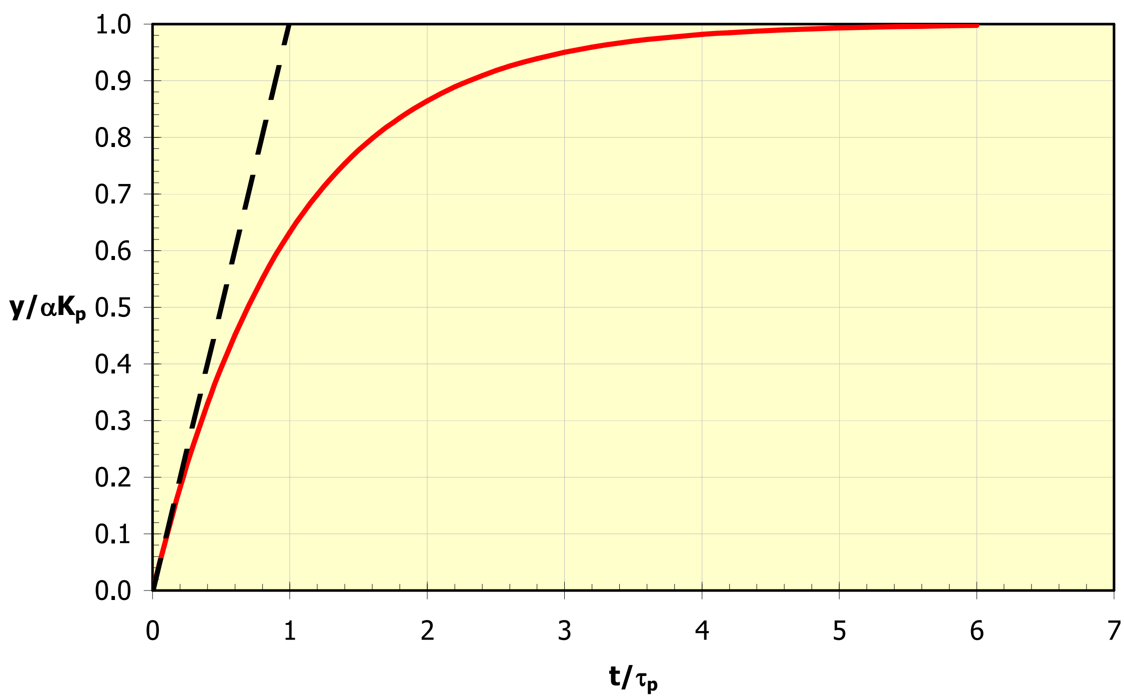
$$= \alpha K_p\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau_p}}\right)$$

$$\therefore \quad y(t) = \alpha K_p\left(1-e^{-t/\tau_p}\right).$$

Notice that K_p is the fraction of the value of the input disturbance that will show up on the output signal. Also notice that the slope is:

$$\frac{dy}{dt} = \frac{\alpha K_p}{\tau_p}e^{-t/\tau_p} \quad \Rightarrow \quad \left.\frac{dy}{dt}\right|_{t=0} = \frac{\alpha K_p}{\tau_p}$$

If the system would maintain its initial rate of change, then it would achieve it ultimate value in one time constant, i.e., when $t=\tau_p$. In reality, the final value is reached in an exponential decay manner — in reality, it takes about $4\tau_p$ to reach the ultimate value (when $y \approx 0.98\alpha K_p$).



If pure integrator:

$$\overline{y}(s) = G(s)\cdot\overline{f}(s) = \frac{K'_p}{s}\frac{\alpha}{s} = \frac{\alpha K'_p}{s^2} \quad \Rightarrow \quad y(t) = \alpha K'_p t$$

This shows the integrating nature of this type of 1st order process.

Ramp. $f(t) = mt \quad \Rightarrow \quad \overline{f}(s) = m/s^2$. So, if 1st order lag:

$$\overline{y}(s) = G(s)\cdot\overline{f}(s) = \frac{K_p}{\tau_p s + 1}\frac{m}{s^2} = mK_p\left(\frac{1}{s^2} - \frac{\tau_p}{s} + \frac{\tau_p^2}{\tau_p s + 1}\right)$$

$$\therefore \quad y(t) = mK_p\left(t - \tau_p + \tau_p e^{-t/\tau_p}\right).$$

At large times, then:

$$y(t) \approx mK_p\left(t - \tau_p\right)$$

Again, K_p is the fraction of the value of the input disturbance that will show up on the output signal. Now τ_p represents a time offset — how far behind the output signal lags behind the input signal.

Sinusoidal Response. $f(t) = \alpha\sin\omega t \quad \Rightarrow \quad \overline{f}(s) = \alpha\omega/(s^2 + \omega^2)$. So, if 1st order lag:

$$\overline{y}(s) = G(s)\cdot\overline{f}(s) = \frac{K_p}{\tau_p s + 1}\frac{\alpha\omega}{s^2 + \omega^2} = \frac{\alpha\omega K_p}{1 + \tau_p^2\omega^2}\left(\frac{\tau_p^2}{\tau_p s + 1} + \frac{1 - \tau_p s}{s^2 + \omega^2}\right)$$

$$y(t) = \frac{\alpha\omega K_p}{1 + \tau_p^2\omega^2}\left(\tau_p e^{-t/\tau_p} + \frac{1}{\omega}\sin\omega t - \tau_p\cos\omega t\right)$$

$$y(t) = \frac{\alpha\omega\tau_p K_p}{1 + \tau_p^2\omega^2}e^{-t/\tau_p} + \frac{\alpha K_p}{1 + \tau_p^2\omega^2}\sin\omega t - \frac{\alpha\omega\tau_p K_p}{1 + \tau_p^2\omega^2}\cos\omega t$$

There is a trigonometric identity:

$$p\cos\theta + q\sin\theta = r\sin(\theta + \phi)$$

where: $r = \sqrt{p^2 + q^2}$

$$\tan\phi = \frac{p}{q}$$

So for this problem:

$$r=\sqrt{\left(\frac{\alpha\omega\tau_p K_p}{1+\tau_p^2\omega^2}\right)^2+\left(\frac{\alpha K_p}{1+\tau_p^2\omega^2}\right)^2}=\frac{\alpha K_p\sqrt{1+\tau_p^2\omega^2}}{1+\tau_p^2\omega^2}=\frac{\alpha K_p}{\sqrt{1+\tau_p^2\omega^2}}$$

$$\tan\phi=\frac{-\dfrac{\alpha\omega\tau_p K_p}{1+\tau_p^2\omega^2}}{\dfrac{\alpha K_p}{1+\tau_p^2\omega^2}}=-\omega\tau_p \quad\Rightarrow\quad \phi=\tan^{-1}\left(\omega\tau_p\right)$$

and:

$$y(t)=\frac{\alpha\omega\tau_p K_p}{1+\tau_p^2\omega^2}e^{-t/\tau_p}+\frac{\alpha K_p}{\sqrt{1+\tau_p^2\omega^2}}\sin\left[\omega t-\tan^{-1}\left(\omega\tau_p\right)\right]$$

At large times, then:

$$y(t)\approx\frac{\alpha K_p}{\sqrt{1+\tau_p^2\omega^2}}\sin\left[\omega t-\tan^{-1}\left(\omega\tau_p\right)\right]$$

Again, K_p represents part of the fraction of the value of the input disturbance that will show up on the output signal. τ_p also plays are a part in the gain, but, more importantly, represents a time offset that manifests itself as a phase angle lag (lag because the angel is subtracted from the input signal). At most, even with very large τ_p values, the output can lag by no more than 90°.

Determination of Coefficients from Data

How could we determine the parameters in the transfer function for a 1st order process? We can perturb the process in a controlled manner & look at the transient results.

Impulse disturbance. $f(t)=\alpha\cdot\delta(0)$ gives:

$$y(t)=\alpha\frac{K_p}{\tau_p}e^{-t/\tau_p}$$

Can put into "linear form" by taking the logarithm:

$$\ln(y) = \ln\left(\frac{\alpha K_p}{\tau_p}\right) - \left(\frac{1}{\tau_p}\right)t$$

Using linear regression of $\ln(y)$ vs. t data, the slope will be $-1/\tau_p$ and the intercept will be $\ln(\alpha K_p/\tau_p)$.

Unit step change. $f(t) = \alpha \cdot H(t)$ gives:

$$y(t) = \alpha K_p \left(1 - e^{-t/\tau_p}\right)$$

with the initial slope of:

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{\alpha K_p}{\tau_p}$$

One would get the gain from the ultimate value, $K_p = y_\infty / \alpha$; however, this requires that you take data long enough to be confident of the ultimate value, y_∞. You could then get the initial slope & determine the time constant:

$$\tau_p = \frac{\alpha K_p}{\left.(dy/dt)\right|_{t=0}}$$

However, then numerical determination of the derivative from data is inherently difficult. You could also put this equation into "linear form:"

$$\ln\left(1 - \frac{y}{\alpha K_p}\right) = -\left(\frac{1}{\tau_p}\right)t$$

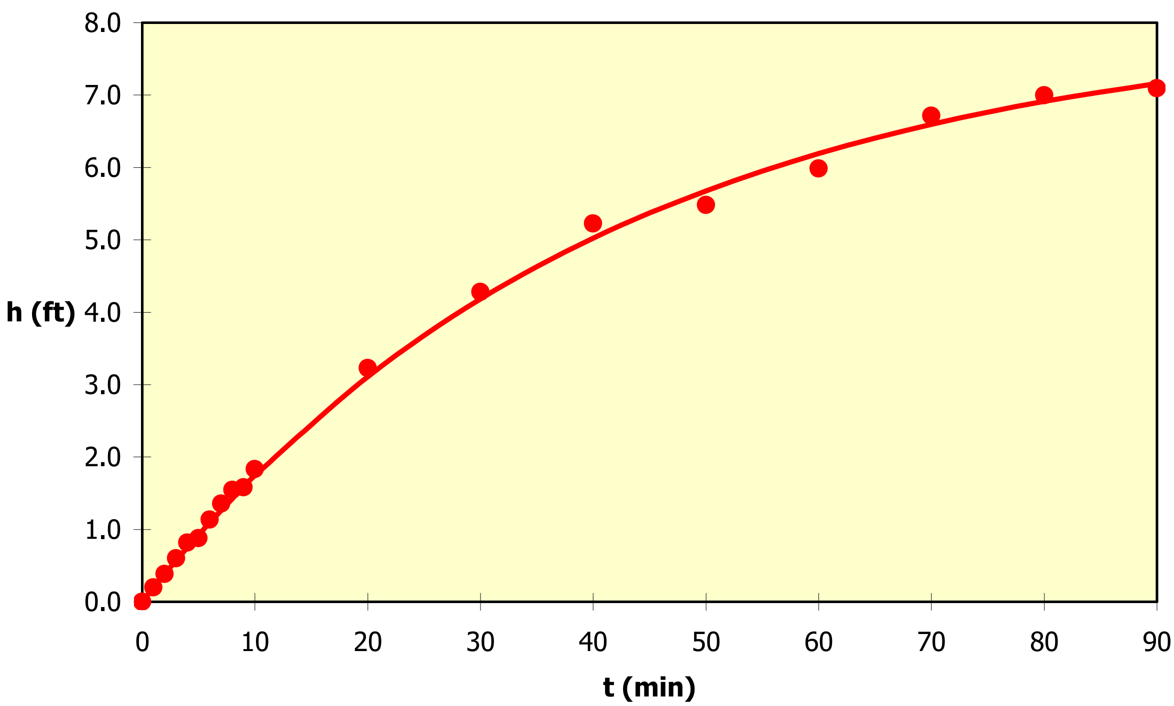
and linear regression can again be used. The difficulty now is that the form of the independent variable, $\ln(1 - y/\alpha K_p)$, includes one of the variables to be determined, K_p. There are two ways this could be handled:

1. Adjust the value of K_p until the intercept from the linear regression is zero.
2. Do the regression by forcing the intercept to be zero & adjust K_p to maximize the regression coefficient, r^2.

The following table shows the response of a tank when the flow rate is increased from 2 ft^3/min to 3 ft^3/min. Using deviation variables, the value of α will be $\alpha = 3 - 2 = 1$ ft^3/min.

min	ft	min	ft
0	0.0	10	1.8
1	0.2	20	3.2
2	0.4	30	4.3
3	0.6	40	5.2
4	0.8	50	5.5
5	0.9	60	6.0
6	1.1	70	6.7
7	1.4	80	7.0
8	1.5	90	7.1
9	1.6		

Doing linear regression with the linear form and allowing an intercept, a value of $K_p = 7.92 \text{ min/ft}^2$ gives an intercept of zero; the corresponding slope gives $\tau_p = 39.0 \text{ min}$. Doing linear regression with the linear form and not allowing an intercept, a value of $K_p = 8.05 \text{ min/ft}^2$ gives an intercept of zero; the corresponding slope gives $\tau_p = 41.0 \text{ min}$. Note that the values were generated with random perturbations added to a linear model with $K_p = 8 \text{ min/ft}^2$ and $\tau_p = 40 \text{ min}$. The following chart shows the



Linearization

Not all dynamic systems are described by linear ODEs — in truth, probably none of the systems are really linear. To use transfer functions & Laplace transforms, however, we must linearize the system of equations. We have already looked at an example of this with flow through a valve. If the flow through a valve is considered to be proportional to $\sqrt{h}$, then the material balance around a tank containing a fluid of constant density is described by:

$$\frac{dV}{dt} = F_0 - F_1 = F_0 - C_v\sqrt{h}$$

and for a constant cross-sectional area in the tank, A, then:

$$A\frac{dh}{dt} = F_0 - C_v\sqrt{h}\,.$$

We've discussed ways to linearize the square root term in this ODE. We can effectively do a Taylor series expansion by first taking the total differential of the right-hand-side:

$$d\left(F_0 - C_v\sqrt{h}\right) = dF_0 - \left(\frac{C_v}{2\sqrt{h}}\right)dh$$

Now, replace the differentials with deviation variables & evaluate any coefficients at the initial steady state. So:

$$\left(F_0 - C_v\sqrt{h}\right)' \approx F_0' - \left(\frac{C_v}{2\sqrt{h^*}}\right)h'$$

and the linearized ODE will be:

$$A\frac{dh'}{dt} = F_0' - \left(\frac{C_v}{2\sqrt{h'}}\right)h'$$

$$A\frac{dh'}{dt} + \left(\frac{C_v}{2\sqrt{h^*}}\right)h' = F_0'$$

$$\left(\frac{2A\sqrt{h^*}}{C_v}\right)\frac{dh'}{dt} + h' = \left(\frac{2\sqrt{h^*}}{C_v}\right)F_0'\,.$$

From this, we see that the linearized ODE is simply a 1st order ODE with the following gain and time constant:

$$K_p = \frac{2\sqrt{h^*}}{C_v}$$

$$\tau_p = \frac{2A\sqrt{h^*}}{C_v}$$

Effectively, we have found that the gain and time constant are not constant, but rather functions of the liquid level. Whether or not the linearized equation is a good representation depends upon how far we perturb the system from its steady state value. For example, given a tank with a cross-sectional area of 5 m^2 which maintains a level of 16 m when the flow in is 2 m^3/min, what happens when we shut off the flow? In the full non-linear solution:

$$\frac{dh}{dt} = F_0 - \frac{C_v}{A}\sqrt{h}$$

where:

$$h(0) = h^* = 16 \text{ m}$$
$$F_0^* = 2 \text{ m}^3/\text{min}$$
$$F_0(t) = 0$$

then:

$$\frac{dh}{\sqrt{h}} = -\frac{C_v}{A}dt \quad \Rightarrow \quad \int_{h_{ss}}^{h} \frac{dh}{\sqrt{h}} = -\frac{C_v}{A}\int_0^t dt \quad \Rightarrow \quad h = \left(\sqrt{h^*} - \frac{C_v}{2A}t\right)^2$$

The linearized equation is:

$$A\frac{dh}{dt} + \frac{C_v\left(h - h^*\right)}{2\sqrt{h^*}} = -F_0^* \quad \Rightarrow \quad A\frac{dh'}{dt} + \frac{C_v}{2\sqrt{h^*}}h' = -F_0^*$$
$$\left(As + \frac{C_v}{2\sqrt{h^*}}\right)\overline{h}' = -\frac{F_0^*}{s}$$
$$\overline{h}' = -\frac{K_p}{\tau_p s + 1}\frac{F_0^*}{s} \text{ where } K_p = \frac{2\sqrt{h^*}}{C_v} \text{ and } \tau_p = \frac{2A\sqrt{h^*}}{C_v}$$

So: $$h' = -F_0^* K_p\left(1 - e^{-t/\tau_p}\right) \quad \Rightarrow \quad h = h^* - \frac{2F_0^*\sqrt{h^*}}{C_v}\left(1 - \exp\left(-\frac{tC_v}{2A\sqrt{h^*}}\right)\right)$$

The chart below shows the difference in the drainage curve for the two equations. Notice that the linearized equation stays fairly close to the exact solution for the 1st 8 ft change of level. Also note that even though the ODE is linearized, it does not predict a straight line answer — there is still curvature to the final $h(t)$ result.