Linear Open Loop Systems

2nd Order Systems

Output modeled with a 2nd order ODE:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bf(t)$$

If $a_0 \neq 0$, then:

$$\frac{a_2}{a_0}\frac{d^2 y}{dt^2} + \frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0} f(t) \quad \Rightarrow \quad \tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = K_p f(t)$$

where: τ is the *natural period of oscillation*.
ζ is the *damping factor*.
K_p is the *steady state gain*.

For deviation variables, where $y(0) = f(0) = 0$, the Laplace transform will be:

$$\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)\overline{y}(s) = K_p \overline{f}(s) \quad \Rightarrow \quad G(s) = \frac{\overline{y}(s)}{\overline{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

2nd order systems arise from several situations:

- Multicapacity processes — two or more 1st order systems in series.
- Inherently 2nd order system — not typical in chemical-type processes.
- Process with controller.

Dynamic Response of 2nd Order System

We can express the transfer function in terms of its roots, or poles:

$$G(s)=\frac{K_p}{\tau^2 s^2+2\zeta\tau s+1}=\frac{K_p/\tau^2}{s^2+\frac{2\zeta}{\tau}s+\frac{1}{\tau^2}}=\frac{K_p/\tau^2}{(s-p_1)(s-p_2)}.$$

What are the poles associated with the 2nd order equation? Can determine the roots using the quadratic equation:

$$p_1,p_2=\frac{-\frac{2\zeta}{\tau}\pm\sqrt{\frac{4\zeta^2}{\tau^2}-\frac{4}{\tau^2}}}{2\tau^2}=-\frac{\zeta}{\tau}\pm\frac{\sqrt{\zeta^2-1}}{\tau}$$

The nature of the solution depends upon the damping factor, ζ:

- *Over Damped Response*. If $|\zeta|>1$, then the poles are real & distinct, leading to the terms in the transfer function:

  $$G(s)=\frac{K_p}{\tau^2}\frac{C_1}{s-p_1}+\frac{K_p}{\tau^2}\frac{C_2}{s-p_2}$$

  with the following terms in the solution:

  $$y(t)=C_1\frac{K_p}{\tau^2}e^{tp_1}+C_2\frac{K_p}{\tau^2}e^{tp_2}$$

- *Critically Damped Response*. If $|\zeta|=1$, then there is a multiple real pole, $p=-1/\tau$, leading to the terms:

  $$G(s)=\frac{K_p}{\tau^2}\frac{C_1\left(s+\frac{1}{\tau}\right)+C_2}{\left(s+\frac{1}{\tau}\right)^2}$$

  with the following terms in the solution:

  $$y(t)=C_1\frac{K_p}{\tau^2}e^{-t/\tau}+C_2\frac{K_p}{\tau^2}te^{-t/\tau}$$

- *Under Damped Response*. If $|\zeta|<1$, then the poles are complex conjugates, leading to the terms in the transfer function:

$$G(s)=\frac{K_p}{\tau^2}\frac{C_1\left(s+\frac{\zeta}{\tau}\right)+C_2}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}$$

with the following terms in the solution:

$$y(t)=C_1\frac{K_p}{\tau^2}e^{-t\zeta/\tau}\cos\left(\frac{t\sqrt{1-\zeta^2}}{\tau}\right)+C_2\frac{K_p}{\tau\sqrt{1-\zeta^2}}e^{-t\zeta/\tau}\sin\left(\frac{t\sqrt{1-\zeta^2}}{\tau}\right)$$

Step Change Response

Let's look at response to a unit step change. $f(t)=\alpha\cdot H(t)\ \Rightarrow\ \bar{f}(s)=\alpha/s$. So, for a 2nd order system:

$$\bar{y}(s)=\frac{K_p}{\tau^2 s^2+2\zeta\tau s+1}\frac{\alpha}{s}=\alpha K_p\left[\frac{1}{s}-\frac{\tau^2 s+2\zeta\tau}{\tau^2 s^2+2\zeta\tau s+1}\right]$$

$$\bar{y}(s)=\alpha K_p\left[\frac{1}{s}-\frac{s+\frac{2\zeta}{\tau}}{s^2+\frac{2\zeta}{\tau}s+\frac{1}{\tau^2}}\right]$$

$$\therefore\quad \bar{y}(s)=\alpha K_p\left[\frac{1}{s}-\frac{\left(s+\frac{\zeta}{\tau}\right)+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2+\frac{1-\zeta^2}{\tau^2}}\right]$$

The form for inverting the 2nd term depends upon the value of the damping factor, ζ. For an over damped 2nd order system, $\zeta>1$, and:

$$\bar{y}(s)=\alpha K_p\left[\frac{1}{s}-\frac{\left(s+\frac{\zeta}{\tau}\right)+\frac{\zeta}{\tau}}{\left(s+\frac{\zeta}{\tau}\right)^2-\frac{\zeta^2-1}{\tau^2}}\right]$$

$$y(t) = \alpha K_p \left[1 - e^{-t\zeta/\tau} \cosh\left(\frac{t\sqrt{\zeta^2 - 1}}{\tau} \right) - \frac{\zeta}{\sqrt{\zeta^2 - 1}} e^{-t\zeta/\tau} \sinh\left(\frac{t\sqrt{\zeta^2 - 1}}{\tau} \right) \right]$$
$$= \alpha K_p \left[1 - e^{-t\zeta/\tau} \left\{ \cosh\left(\frac{t\sqrt{\zeta^2 - 1}}{\tau} \right) + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh\left(\frac{t\sqrt{\zeta^2 - 1}}{\tau} \right) \right\} \right]$$

This type of response is termed *over damped* since it resembles a sluggish exponential decay. It becomes more and more sluggish as ζ is increased.

Next, for a critically damped 2nd order system, $\zeta = 1$:

$$\bar{y}(s) = \alpha K_p \left[\frac{1}{s} - \frac{\left(s + \frac{1}{\tau} \right) + \frac{1}{\tau}}{\left(s + \frac{1}{\tau} \right)^2} \right] = \alpha K_p \left[\frac{1}{s} - \frac{1}{\left(s + \frac{1}{\tau} \right)} - \frac{1}{\tau} \frac{1}{\left(s + \frac{1}{\tau} \right)^2} \right]$$

$$y(t) = \alpha K_p \left[1 - \left(1 + \frac{t}{\tau} \right) e^{-t/\tau} \right]$$

This is called critically damped because this is the smallest value of ζ to retain purely exponential decay behavior.

Now, for an under damped 2nd order system, $\zeta < 1$:

$$\bar{y}(s) = \alpha K_p \left[\frac{1}{s} - \frac{\left(s + \frac{\zeta}{\tau} \right) + \frac{\zeta}{\tau}}{\left(s + \frac{\zeta}{\tau} \right)^2 + \frac{1 - \zeta^2}{\tau^2}} \right]$$

$$y(t) = \alpha K_p \left[1 - e^{-t\zeta/\tau} \cos\left(\frac{t\sqrt{1 - \zeta^2}}{\tau} \right) - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-t\zeta/\tau} \sin\left(\frac{t\sqrt{1 - \zeta^2}}{\tau} \right) \right]$$
$$= \alpha K_p \left[1 - e^{-t\zeta/\tau} \left\{ \cos\left(\frac{t\sqrt{1 - \zeta^2}}{\tau} \right) + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin\left(\frac{t\sqrt{1 - \zeta^2}}{\tau} \right) \right\} \right]$$

Notice that this response shows an exponential decay but there is also oscillatory behavior. We can better see what the oscillatory behavior looks like if we combine the sine & cosine terms into a single sine term with a phase angle offset. Remember:

$$p\cos\theta + q\sin\theta = r\sin(\theta+\phi)$$

where: $r = \sqrt{p^2 + q^2}$

$$\tan\phi = \frac{p}{q}$$

So:

$$r = \sqrt{1 + \frac{\zeta^2}{1-\zeta^2}} = \frac{1}{\sqrt{1-\zeta^2}}$$

$$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

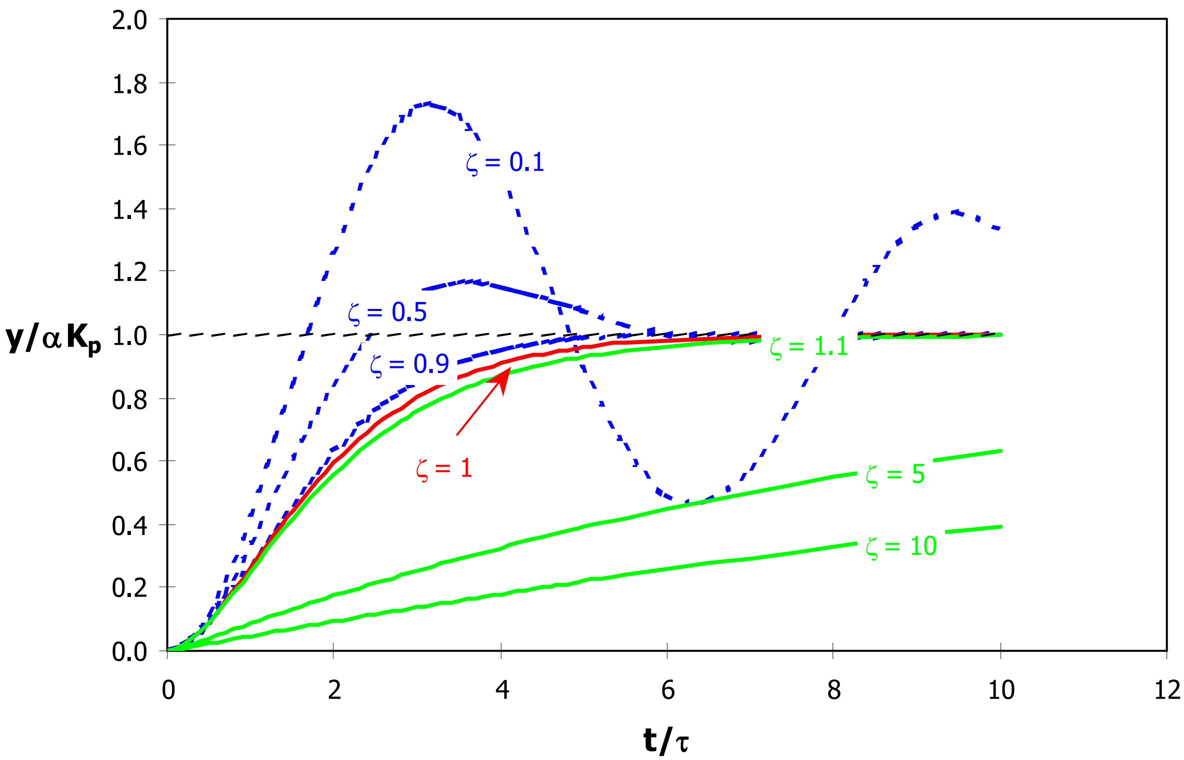
Defining:

$$\omega \equiv \frac{\sqrt{1-\zeta^2}}{\tau}$$

then:

$$y(t) = \alpha K_p \left[1 - \frac{e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t + \phi) \right]$$

The following figure shows how the response to a unit step change of a 2nd order system depending upon the damping factor. Note that all curves have an initial slope of zero — this is different from a 1st order system. Note that as ζ gets smaller and smaller, the oscillations get larger and take longer to damp out.



Descriptors for an Underdamped System

There are several terms defined to describe the characteristics of an underdamped system's response. These are also shown in the following figure:

- *Overshoot*. A measure of how far the response exceeds the ultimate value. From the figure, this is A/α. From the response curve, the value is:

$$\text{Overshoot} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

- *Decay Ratio*. The ratio of the overshoot of two successive peaks, C/A. It can be shown that this is:

$$\text{Decay Ratio} = (\text{Overshoot})^2 = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

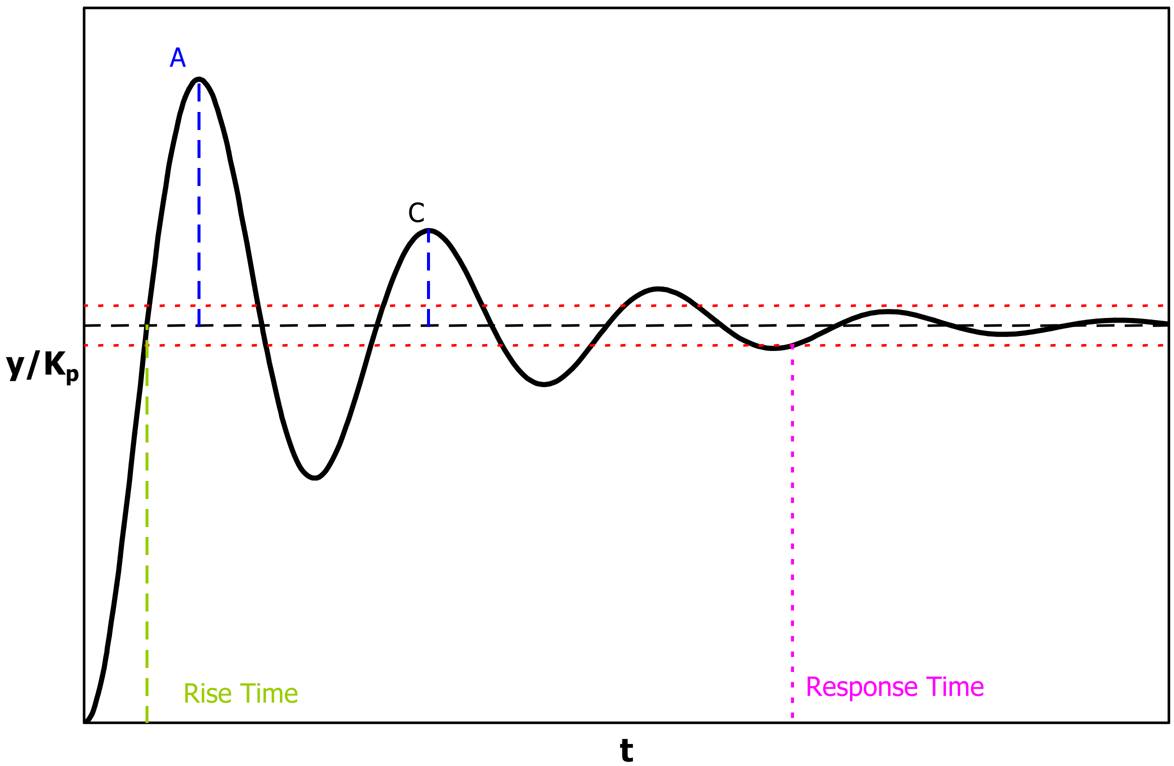
- *Period of oscillation*. The time between peaks. Since the frequency of oscillation is:

$$\omega = \frac{\sqrt{1-\zeta^2}}{\tau}$$

then the period of oscillation is:

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

- *Rise time*. The time it takes the response to 1st get to the ultimate value.
- *Response time*. The time it takes the response to stay within 5% of the ultimate value.

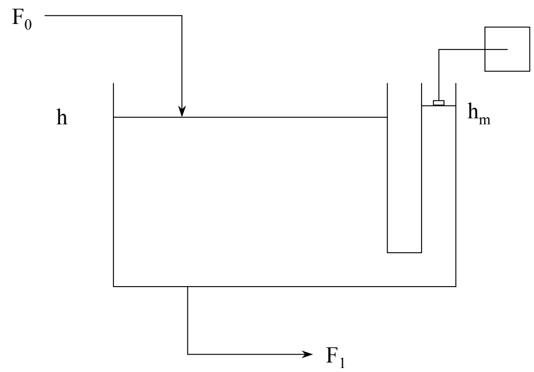


Natural 2nd Order Systems

There are four natural 2nd order systems described in Appendix 11A of Stephanopolous's text (pp. 205 - 211):

- Simple manometers
- Externally mounted level indicators
- Variable capacitance differential pressure transducers
- Pneumatic valves

We will go through the derivation of the transfer function for the externally mounted level indicators since it is not given in the text.



We want to look at the flow of fluid within the external tube of length L having constant cross-sectional area A_m. Using a similar derivation as that done to get Equation 11A.5, let us look at the forces along the plane at the bottom of the displacement chamber:

$$\mathsf{F}_{\text{tank}} - \mathsf{F}_{\text{tube}} - \mathsf{F}_{\text{friction}} = \frac{m_{\text{tube}}}{g_c} a_{\text{tube}}$$

where:

- F_{tank} force at the entrance of the displacement tube due to the static head in the tank.
- F_{tube} force at the bottom of the displacement tube due to the static head in the displacement tube.
- $\mathsf{F}_{\text{friction}}$ frictional losses in the displacement tube.
- m_{tube} mass of fluid in the tube.
- a_{tube} acceleration of the fluid in the tube.

As in the text, if we assume laminar flow in the displacement tube, then:

$$\mathsf{F}_{\text{friction}} = A_m \frac{8\mu L}{R^2 g_c} v_{ave} = A_m \frac{8\mu L}{R^2 g_c} \frac{dh_m}{dt}$$

The acceleration of the fluid in the tube is:

$$a_{\text{tube}} = \frac{dv_{ave}}{dt} = \frac{d^2 h_m}{dt^2}$$

so:

$$\rho\frac{g}{g_c}A_m h-\rho\frac{g}{g_c}A_m h_m - A_m\frac{8\mu L}{R^2 g_c}\frac{dh_m}{dt}=\frac{\rho A_m L}{g_c}\frac{d^2 h_m}{dt^2}$$

$$\frac{\rho L}{g_c}\frac{d^2 h_m}{dt^2}+\frac{8\mu L}{R^2 g_c}\frac{dh_m}{dt}+\frac{\rho g}{g_c}h_m=\frac{\rho g}{g_c}h$$

$$\frac{L}{g}\frac{d^2 h_m}{dt^2}+\frac{8\mu L}{R^2\rho g}\frac{dh_m}{dt}+h_m=h$$

Putting this ODE into the standard 2nd order form gives:

$$\tau^2=\frac{L}{g} \quad\Rightarrow\quad \tau=\sqrt{\frac{L}{g}}$$

$$2\tau\zeta=\frac{8\mu L}{R^2\rho g} \quad\Rightarrow\quad \zeta=\frac{4\mu L}{R^2\rho g}\cdot\frac{1}{\tau}=\frac{8\mu L}{R^2\rho g}\sqrt{\frac{g}{L}}=\frac{8\mu}{\rho R^2}\sqrt{\frac{L}{g}}$$

and: $K_p=1$