

## Underdamped Systems

### 2<sup>nd</sup> Order Systems

Output modeled with a 2<sup>nd</sup> order ODE:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b \cdot f(t)$$

If  $a_0 \neq 0$ , then:

$$\frac{a_2}{a_0} \frac{d^2 y}{dt^2} + \frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} f(t) \Rightarrow \tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = K_p f(t)$$

where:  $\tau$  is the *natural period of oscillation*.

$\zeta$  is the *damping factor*.

$K_p$  is the *steady state gain*.

For deviation variables, where  $y(0) = f(0) = 0$ , the Laplace transform will be:

$$(\tau^2 s^2 + 2\zeta\tau s + 1)\bar{y}(s) = K_p \bar{f}(s) \Rightarrow G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

### Dynamic Response of Underdamped 2<sup>nd</sup> Order System

If  $|\zeta| < 1$ , then the poles in the transfer function are complex conjugates. Let's look at response to a unit step change.  $f(t) = \alpha \cdot H(t) \Rightarrow \bar{f}(s) = \alpha/s$ . So, for a 2<sup>nd</sup> order system:

$$\bar{y}(s) = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1} \cdot \frac{\alpha}{s}$$

$$\bar{y}(s) = \alpha K_p \left[ \frac{1}{s} - \frac{\tau^2 s + 2\zeta\tau}{\tau^2 s^2 + 2\zeta\tau s + 1} \right]$$

$$\bar{y}(s) = \alpha K_p \left[ \frac{1}{s} - \frac{s + \frac{2\zeta}{\tau}}{s^2 + \frac{2\zeta}{\tau}s + \frac{1}{\tau^2}} \right]$$

$$\therefore \bar{y}(s) = \alpha K_p \left[ \frac{1}{s} - \frac{\left(s + \frac{\zeta}{\tau}\right) + \frac{\zeta}{\tau}}{\left(s + \frac{\zeta}{\tau}\right)^2 + \frac{1 - \zeta^2}{\tau^2}} \right]$$

Inverting:

$$y(t) = \alpha K_p \left[ 1 - e^{-t\zeta/\tau} \cos\left(\frac{t\sqrt{1-\zeta^2}}{\tau}\right) - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-t\zeta/\tau} \sin\left(\frac{t\sqrt{1-\zeta^2}}{\tau}\right) \right]$$

The sine & cosine terms can be combined into a single sine term with a phase angle offset.  
Remember:

$$p \cos \theta + q \sin \theta = r \sin(\theta + \phi)$$

where:  $r = \sqrt{p^2 + q^2}$  and  $\tan \phi = \frac{p}{q}$

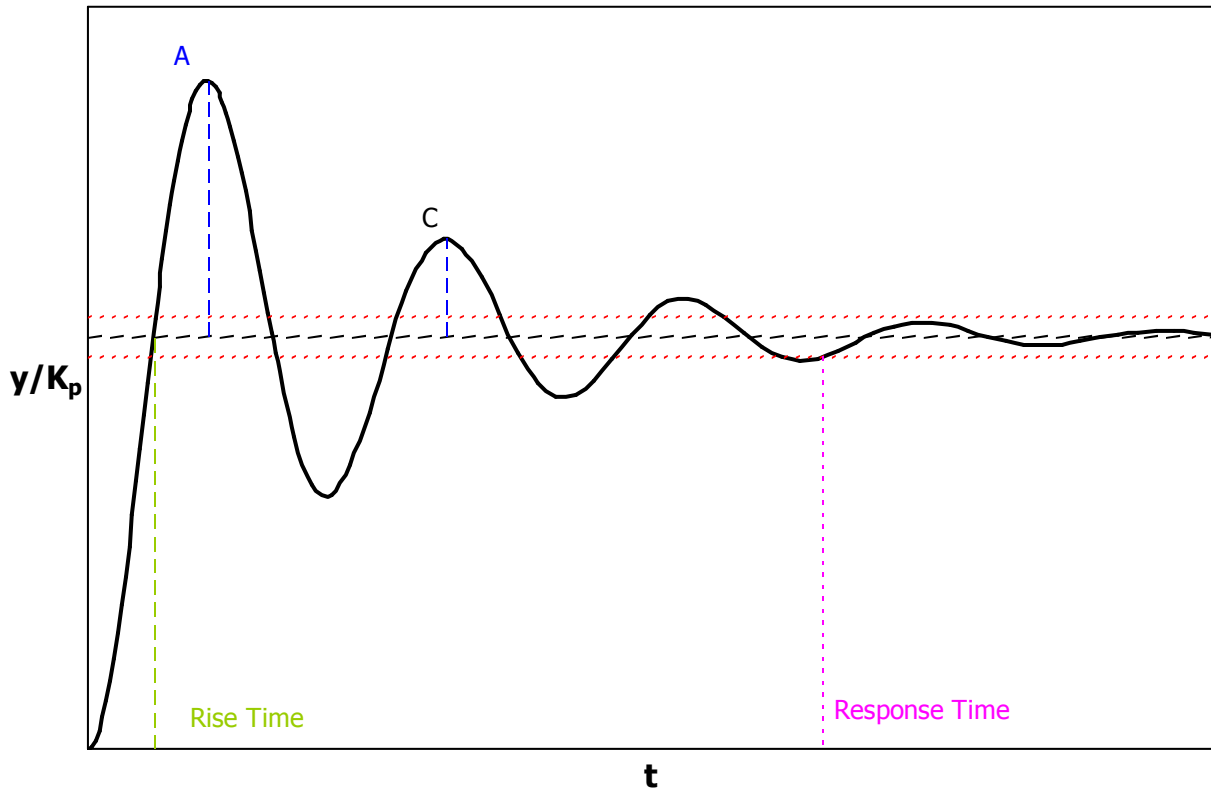
So:  $r = \sqrt{1 + \frac{\zeta^2}{1-\zeta^2}} = \frac{1}{\sqrt{1-\zeta^2}}$

$$\tan \phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Defining:

$$\omega \equiv \frac{\sqrt{1-\zeta^2}}{\tau}$$

then:  $y(t) = \alpha K_p \left[ 1 - \frac{e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t + \phi) \right]$



**Figure 1. Response Curve for Underdamped System**

There are several terms defined to describe the characteristics of an underdamped system's response. These are also shown in the Figure 1:

- *Ultimate Value.* This is the value that the response settles down to at very large times. This can easily be determined as:

$$y_{\infty} = \lim_{t \rightarrow \infty} \{y(t)\} = \lim_{t \rightarrow \infty} \left\{ \alpha K_p \left[ 1 - \frac{e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t + \phi) \right] \right\} = \alpha K_p$$

- *Period of oscillation.* The time between crossing of the ultimate value. This will also be the time between the peaks and valleys. Since the frequency of oscillation is:

$$\omega = \frac{\sqrt{1-\zeta^2}}{\tau}$$

then the period of oscillation is:

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

- *Rise time.* The time it takes the response to first get to the ultimate value. This can be easily determined as:

$$y(t_{\text{rise}}) = \alpha K_p \left[ 1 - \frac{e^{-t_{\text{rise}}\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t_{\text{rise}} + \phi) \right] = \alpha K_p$$

$$1 - \frac{e^{-t_{\text{rise}}\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t_{\text{rise}} + \phi) = 1$$

$$\frac{e^{-t_{\text{rise}}\zeta/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega t_{\text{rise}} + \phi) = 0$$

$$\sin(\omega t_{\text{rise}} + \phi) = 0$$

$$\omega t_{\text{rise}} + \phi = \pi$$

$$\therefore t_{\text{rise}} = \frac{\pi - \phi}{\omega} = \frac{\tau}{\sqrt{1-\zeta^2}} \left[ \pi - \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \right]$$

- *Overshoot.* A measure of how far the response exceeds the ultimate value. On the figure, this is  $A/\alpha$ . The formula given for this in the text is:

$$\text{Overshoot} = \exp \left( \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \right)$$

This is calculated by finding the time at which the first maximum occurs and then finding the corresponding value in the response curve. The 1<sup>st</sup> derivative of the response curve is:

$$\frac{dy}{dt} = -\frac{\alpha K_p}{\sqrt{1-\zeta^2}} \frac{d}{dt} \left[ e^{-t\zeta/\tau} \sin(\omega t + \phi) \right]$$

$$\frac{dy}{dt} = -\frac{\alpha K_p}{\sqrt{1-\zeta^2}} \left[ -\frac{\zeta}{\tau} e^{-t\zeta/\tau} \sin(\omega t + \phi) + \omega e^{-t\zeta/\tau} \cos(\omega t + \phi) \right]$$

$$\frac{dy}{dt} = -\frac{\alpha K_p e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \left[ -\frac{\zeta}{\tau} \sin(\omega t + \phi) + \omega \cos(\omega t + \phi) \right]$$

The sine & cosine terms can be combined into a single sine term to give:

$$\frac{dy}{dt} = -\frac{\alpha K_p e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \sqrt{\omega^2 + \frac{\zeta^2}{\tau^2}} \sin \left[ \omega t + \phi + \tan^{-1} \left( -\frac{\omega\tau}{\zeta} \right) \right]$$

$$\frac{dy}{dt} = -\frac{\alpha K_p e^{-t\zeta/\tau}}{\sqrt{1-\zeta^2}} \sqrt{\frac{1-\zeta^2}{\tau^2} + \frac{\zeta^2}{\tau^2}} \sin \left[ \omega t + \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) - \tan^{-1} \left( \frac{\tau \sqrt{1-\zeta^2}}{\zeta} \right) \right]$$

$$\therefore \frac{dy}{dt} = -\frac{\alpha K_p e^{-t\zeta/\tau}}{\tau \sqrt{1-\zeta^2}} \sin(\omega t)$$

So, the time  $t^*$  at which the 1<sup>st</sup> maximum occurs is at:

$$\left. \frac{dy}{dt} \right|_{t=t^*} = -\frac{\alpha K_p e^{-t^*\zeta/\tau}}{\tau \sqrt{1-\zeta^2}} \sin(\omega t^*) = 0 \Rightarrow \sin(\omega t^*) = 0$$

$$\therefore t^* = \frac{\pi}{\omega}$$

Then:

$$y_{\max} = y(t^*) = \alpha K_p \left[ 1 - \frac{e^{-\pi\zeta/\omega\tau}}{\sqrt{1-\zeta^2}} \sin(\pi + \phi) \right]$$

$$y_{\max} = \alpha K_p \left[ 1 + \frac{1}{\sqrt{1-\zeta^2}} \sin(\phi) \exp\left(-\frac{\pi\zeta}{\omega\tau}\right) \right]$$

$$y_{\max} = \alpha K_p \left[ 1 + \frac{1}{\sqrt{1-\zeta^2}} \sin\left(\tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right) \exp\left(-\frac{\pi\zeta}{\tau \sqrt{1-\zeta^2}}\right) \right]$$

$$y_{\max} = \alpha K_p \left[ 1 + \frac{1}{\sqrt{1-\zeta^2}} \left(\sqrt{1-\zeta^2}\right) \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

$$\therefore y_{\max} = \alpha K_p \left[ 1 + \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

Finally,

$$\text{Overshoot} = \frac{y_{\max} - y_{\infty}}{y_{\infty}} = \frac{\alpha K_p \left[ 1 + \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right] - \alpha K_p}{\alpha K_p} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

- *Decay Ratio.* The ratio of the overshoot of two successive peaks,  $C/A$ . The text gives this expression as:

$$\text{Decay Ratio} = (\text{Overshoot})^2 = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

We know from the derivation for the overshoot that the peaks and valleys will occur when:

$$\sin(\omega t^*) = 0 \Rightarrow \omega t^* = 0, \pi, 2\pi, \dots, n\pi, \dots$$

The peaks will occur at the odd multiples:

$$\omega t^* = \pi, 3\pi, \dots, (2n-1)\pi, \dots$$

So, the n-th peak will have a response value of:

$$y_{\text{peak},n} = y(t_n^*) = \alpha K_p \left[ 1 - \frac{e^{-(2n-1)\pi\zeta/\omega\tau}}{\sqrt{1-\zeta^2}} \sin((2n-1)\pi + \phi) \right]$$

$$y_{\text{peak},n} = \alpha K_p \left[ 1 - \frac{1}{\sqrt{1-\zeta^2}} \sin(\pi + \phi) \exp\left(-\frac{(2n-1)\pi\zeta}{\omega\tau}\right) \right]$$

$$y_{\text{peak},n} = \alpha K_p \left[ 1 + \frac{1}{\sqrt{1-\zeta^2}} \sin(\phi) \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

$$y_{\text{peak},n} = \alpha K_p \left[ 1 + \frac{1}{\sqrt{1-\zeta^2}} \sin\left(\tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right) \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

$$\therefore y_{\text{peak},n} = \alpha K_p \left[ 1 + \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

Note that the valleys will occur at the even multiples:

$$\omega t^* = 2\pi, 4\pi, \dots, 2n\pi, \dots$$

so the n-th valley will have a response value of:

$$y_{\text{valley},n} = \alpha K_p \left[ 1 - \frac{e^{-2n\pi\zeta/\omega\tau}}{\sqrt{1-\zeta^2}} \sin(2n\pi + \phi) \right]$$

$$y_{\text{valley},n} = \alpha K_p \left[ 1 - \frac{1}{\sqrt{1-\zeta^2}} \sin(\phi) \exp\left(-\frac{2n\pi\zeta}{\omega\tau}\right) \right]$$

$$\therefore y_{\text{valley},n} = \alpha K_p \left[ 1 - \exp\left(-\frac{2n\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right]$$

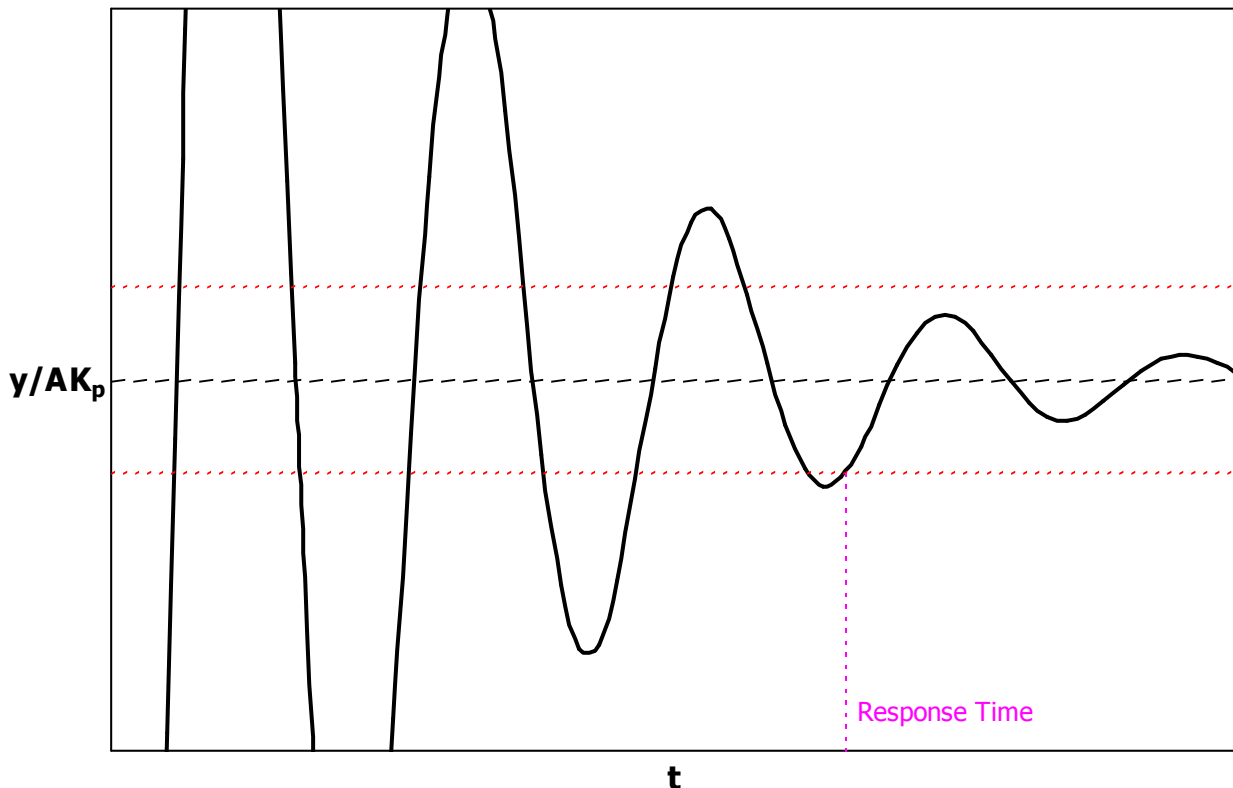
Now, the decay ratio,  $R$ , will be:

$$R = \frac{y_{\text{peak},n+1} - y_{\infty}}{y_{\text{peak},n} - y_{\infty}} = \frac{\alpha K_p \exp\left(-\frac{(2n+1)\pi\zeta}{\sqrt{1-\zeta^2}}\right)}{\alpha K_p \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$R = \frac{\exp\left(-\frac{(2n+1)\pi\zeta}{\sqrt{1-\zeta^2}}\right)}{\exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$R = \exp\left(-\frac{(2n+1)\pi\zeta}{\sqrt{1-\zeta^2}} + \frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\therefore R = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$



**Figure 2. Occurrence of Response Time in an Underdamped System**

- *Response time.* The time it takes the response to stay within  $\pm 5\%$  of the ultimate value. Figure 2 shows that this is a much more complicated value to figure out,

since the response curve may cross over the threshold values many times before it finally settles down within the range. But we can bracket the value by finding the 1<sup>st</sup> peak and the 1<sup>st</sup> valley that remains within the tolerance. If a peak is the 1<sup>st</sup> extremum to stay within the tolerance, then the response time will be between this 1<sup>st</sup> peak within the tolerance and the last valley out of the tolerance.

Similarly, if a valley is the 1<sup>st</sup> extremum to stay within the tolerance, then the response time will be between this 1<sup>st</sup> valley within the tolerance and the last peak out of the tolerance.

Let us denote the tolerance as  $\varepsilon$ , where by tradition  $\varepsilon = 0.05$ . Then the response time will be before the lowest value of  $n$  which satisfies:

$$\begin{aligned}
 y_{\text{peak},n} - y_{\infty} &< \varepsilon y_{\infty} \\
 y_{\text{peak},n} &< (1 + \varepsilon) y_{\infty} \\
 \alpha K_p \left[ 1 + \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right] &< (1 + \varepsilon) \alpha K_p \\
 1 + \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) &< 1 + \varepsilon \\
 \exp\left(-\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}}\right) &< \varepsilon \\
 -\frac{(2n-1)\pi\zeta}{\sqrt{1-\zeta^2}} &< \ln(\varepsilon) \\
 \therefore n_p = \min \left[ n > \frac{1}{2} - \frac{\ln(\varepsilon)\sqrt{1-\zeta^2}}{2\pi\zeta} \right]
 \end{aligned}$$

The response time will also be before the lowest value of  $n$  which satisfies:

$$\begin{aligned}
 y_{\infty} - y_{\text{valley},n} &< \varepsilon y_{\infty} \\
 y_{\text{valley},n} &> (1 - \varepsilon) y_{\infty} \\
 \alpha K_p \left[ 1 - \exp\left(-\frac{2n\pi\zeta}{\sqrt{1-\zeta^2}}\right) \right] &> (1 - \varepsilon) \alpha K_p \\
 \exp\left(-\frac{2n\pi\zeta}{\sqrt{1-\zeta^2}}\right) &< \varepsilon
 \end{aligned}$$

$$-\frac{2n\pi\zeta}{\sqrt{1-\zeta^2}} < \ln(\varepsilon)$$

$$\therefore n_v = \min \left[ n > -\frac{\ln(\varepsilon)\sqrt{1-\zeta^2}}{2\pi\zeta} \right]$$

So, if  $n_p \leq n_v$ , then the response time will be between the peaks represented by  $n_p$  and  $n_v - 1$ , or times  $(2n_p - 1)\pi/\omega$  and  $2(n_v - 1)\pi/\omega$ . However, if  $n_v < n_p$ , then the response time will be between the peaks represented by  $n_v$  and  $n_p - 1$ , or times  $2n_v\pi/\omega$  and  $(2n_p - 3)\pi/\omega$ .

For example, Figure 1 was generated using  $\zeta = 0.15$ . So:

$$-\frac{\ln(\varepsilon)\sqrt{1-\zeta^2}}{2\pi\zeta} = 3.1 \Rightarrow n_v = 4$$

$$\frac{1}{2} - \frac{\ln(\varepsilon)\sqrt{1-\zeta^2}}{2\pi\zeta} = 3.6 \Rightarrow n_p = 4$$

So, the response time is found from where the response curve crosses the lower limit between the 3<sup>rd</sup> valley and the 4<sup>th</sup> peak.

One further thing to note is that the response can be before the rise time if the 1<sup>st</sup> peak does not exceed the tolerance for the response time. This will happen if  $\zeta$  stays larger than a threshold value of:

$$\frac{1}{2} - \frac{\ln(\varepsilon)\sqrt{1-\zeta^{*2}}}{2\pi\zeta^*} = 1 \Rightarrow \zeta^* = \left\{ \left[ \frac{\pi}{\ln(\varepsilon)} \right]^2 + 1 \right\}^{-1/2}$$

For the traditional tolerance, this threshold value is  $\zeta^* = 0.690$ .