Higher Order Systems

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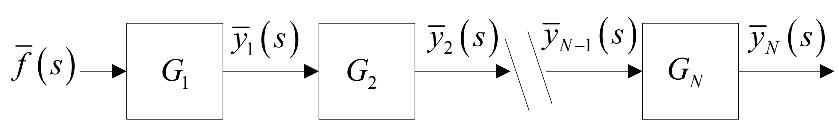
Three types of common systems with higher than 2nd order dynamics are:

- Multicapacity processes in series.
- Processes with dead time.
- Processes with inverse response.

Multicapacity Processes in Series

When we introduced transfer functions, we talked about the overall transfer function for a series of <u>non-interacting</u> processes.



If there are a series of transfer functions, then:

$$
\begin{aligned}
\bar{y}_N(s) &= G_N(s)\,\bar{y}_{N-1}(s) \\
&= G_N(s)\,G_{N-1}(s)\,\bar{y}_{N-2}(s) \\
&= G_N(s)\,G_{N-1}(s)\cdots G_1(s)\,\bar{f}(s)
\end{aligned}
$$

$$\therefore \quad \frac{\bar{y}_N(s)}{\bar{f}(s)} = \prod_{i=1}^{N} G_i(s)$$

Two Non-Interacting 1st Order Systems

For a series of two non-interacting 1st order systems with different time constants τ_1 & τ_2, then the overall transfer function is 2nd order:

$$\frac{\bar{y}_2(s)}{\bar{f}(s)} = \frac{K_1}{\tau_1 s + 1} \cdot \frac{K_2}{\tau_2 s + 1}.$$

Since the denominator of the transfer function can be factored into linear terms each with real coefficients, the overall transfer function will give an overdamped response to a disturbance. Also, since the poles of the transfer function, $-1/\tau_1$ & $-1/\tau_2$, are real and negative, then the overall transfer function will give a <u>stable</u> overdamped response to a disturbance.

We could also analyze the expected response knowing something about 2nd order systems. Putting the overall transfer function into the standard form for a 2nd order system gives:

$$\frac{\bar{y}_2(s)}{\bar{f}(s)} = \frac{K_1}{\tau_1 s + 1} \cdot \frac{K_2}{\tau_2 s + 1} = \frac{K_1 K_2}{\tau_1 \tau_2 s + (\tau_1 + \tau_2) s + 1} \equiv \frac{K_p^*}{\tau_p^{*2} s + 2\zeta_p^* \tau_p^* s + 1}$$

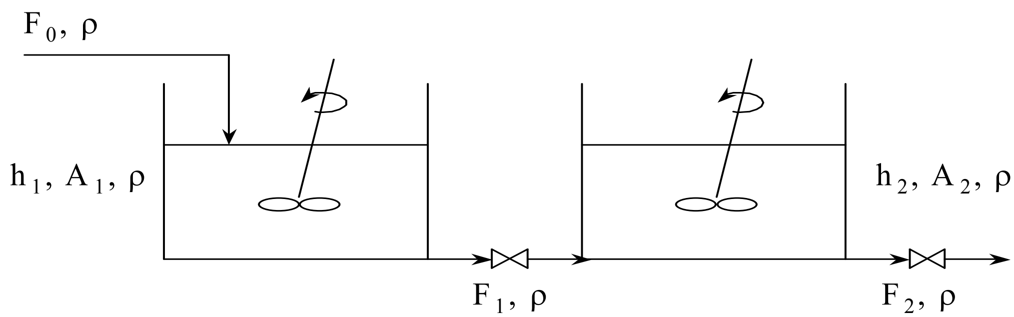
where:

$$\begin{aligned}
&K_p^* = K_1 K_2 \\
&\tau_p^{*2} = \tau_1 \tau_2 \quad \Rightarrow \quad \tau_p^* = \sqrt{\tau_1 \tau_2} \\
&2\zeta_p^* \tau_p^* = \tau_1 + \tau_2 \quad \Rightarrow \quad \zeta_p^* = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}} = \frac{1}{2}\left(\sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}} \right)
\end{aligned}$$

For positive τ_1 & τ_2 that are not equal, then the damping factor ζ_p^* will always be greater than one (meaning an overdamped system). In the event that $\tau_1 = \tau_2$, then $\zeta_p^* = 1$ and the system will be critically damped. There are no combinations of τ_1 & τ_2 so that the system would be underdamped – this makes sense since an underdamped system requires linear factors with complex coefficients.

Two Interacting 1st Order Systems

But what if the processes interact? Can we get an underdamped system then? The answer is – it depends how they interact. Let's look at a simple liquid flow system where the interaction is due to back-pressure from the second tank. Assume the liquid density is constant. Further assume that the tanks are open to the atmosphere, have constant cross-sectional areas A_1 and A_2, and that the flow through each valve is proportional to the liquid level providing the hydrostatic pressure (i.e., $F_i = C_i(-\Delta h)$).



The material balance for Tank 1 leads to:

$$A_1 \frac{dh_1}{dt} = F_0 - C_1(h_1 - h_2)$$
$$A_1 \frac{dh_1}{dt} + C_1 h_1 = F_0 + C_1 h_2$$
$$\frac{A_1}{C_1}\frac{dh_1}{dt} + h_1 = \frac{1}{C_1}F_0 + h_2$$

Since a linear system, can immediately use deviation variables:

$$\frac{A_1}{C_1}\frac{dh_1'}{dt} + h_1' = \frac{1}{C_1}F_0' + h_2'$$
$$\left(\frac{A_1}{C_1}s + 1\right)\overline{h}_1' = \frac{1}{C_1}\overline{F}_0' + \overline{h}_2'$$
$$\overline{h}_1' = \frac{K_1}{\tau_1 s + 1}\overline{F}_0' + \frac{1}{\tau_1 s + 1}\overline{h}_2' \text{ where } \tau_1 = \frac{A_1}{C_1} \text{ and } K_1 = \frac{1}{C_1}.$$

The material balance for Tank 2 leads to:

$$A_2 \frac{dh_2}{dt} = C_1(h_1 - h_2) - C_2 h_2$$
$$A_2 \frac{dh_2}{dt} + (C_1 + C_2)h_2 = C_1 h_1$$

Again, since a linear system, we can immediately use deviation variables:

$$A_2 \frac{dh_2'}{dt} + \left(C_1 + C_2\right) h_2' = C_1 h_1'$$
$$\left(A_2 s + C_1 + C_2\right) \overline{h}_2' = C_1 \overline{h}_1'$$

We would normally define the τ_2 time constant by dividing by the sum of the valve coefficients, $C_1 + C_2$. However, instead, let's define the time constant in a manner similar to that for the 1st tank, i.e., $\tau_2 = A_2 / C_2$. This leads to:

$$\left(\tau_2 C_2 s + C_1 + C_2\right) \overline{h}_2' = C_1 \overline{h}_1'$$

$$\left(\tau_2 s + 1 + \frac{C_1}{C_2}\right) \overline{h}_2' = \frac{C_1}{C_2} \overline{h}_1'$$

so: $\overline{h}_2' = \dfrac{K_2}{\tau_2 s + 1 + K_2} \overline{h}_1'$ where $K_2 = \dfrac{C_1}{C_2}$.

Now, the transfer function between $\overline{F}_0'$ and $\overline{h}_2'$ will be:

$$\overline{h}_2' = \frac{K_2}{\tau_2 s + 1 + K_2} \left(\frac{K_1}{\tau_1 s + 1} \overline{F}_0' + \frac{1}{\tau_1 s + 1} \overline{h}_2' \right)$$

$$\left(1 - \frac{K_2}{\tau_2 s + 1 + K_2} \frac{1}{\tau_1 s + 1}\right) \overline{h}_2' = \frac{K_1}{\tau_1 s + 1} \cdot \frac{K_2}{\tau_2 s + 1 + K_2} \overline{F}_0'$$

$$\frac{\left(\tau_1 s + 1\right)\left(\tau_2 s + 1 + K_2\right) - K_2}{\left(\tau_1 s + 1\right)\left(\tau_2 s + 1 + K_2\right)} \overline{h}_2' = \frac{K_1 K_2}{\left(\tau_1 s + 1\right)\left(\tau_2 s + 1 + K_2\right)} \overline{F}_0'$$

$$\overline{h}_2' = \frac{K_1 K_2}{\left(\tau_1 s + 1\right)\left(\tau_2 s + 1 + K_2\right) - K_2} \overline{F}_0'$$

$$\overline{h}_2' = \frac{K_1 K_2}{\tau_1 \tau_2 s^2 + \left[\tau_1 \left(1 + K_2\right) + \tau_2\right] s + 1} \overline{F}_0'$$

Notice the extra K_2 term in the denominator because of the interaction of the tanks. The coefficients in the denominator are all positive & real, so the resulting response to a disturbance will be stable. But will it be overdamped, critically damped, or underdamped?

To answer this, let's look at the damping factor. In the standard form for a 2nd order system:

$$\tau_p^{*2} = \tau_1\tau_2 \quad \Rightarrow \quad \tau_p^* = \sqrt{\tau_1\tau_2}$$

$$\begin{aligned} 2\zeta_p^*\tau_p^* = \tau_1(1+K_2)+\tau_2 \quad \Rightarrow \quad \zeta_p^* &= \frac{\tau_1(1+K_2)+\tau_2}{2\sqrt{\tau_1\tau_2}} \\ &= \frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}} + \frac{\tau_1 K_2}{2\sqrt{\tau_1\tau_2}} \\ &= \frac{1}{2}\left(\sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}}\right) + \frac{K_2}{2}\sqrt{\frac{\tau_1}{\tau_2}} \end{aligned}$$

Note that the first term for ζ_p^* will always be greater or equal to one (only equal to one if $\tau_1 = \tau_2$). The second term will always be positive. Together, this shows that the overall damping factor will always be greater than one – this type of system will <u>always</u> be overdamped.

We can do an analysis of the poles, too. The poles are given by:

$$p_1, p_2 = \frac{-\left[\tau_1(1+K_2)+\tau_2\right] \pm \sqrt{\left[\tau_1(1+K_2)+\tau_2\right]^2 - 4\tau_1\tau_2}}{2\tau_1\tau_2}$$

$$p_1, p_2 = \frac{-\left[\tau_1(1+K_2)+\tau_2\right] \pm \sqrt{\tau_1^2(1+K_2)^2 + 2\tau_1\tau_2(1+K_2) + \tau_2^2 - 4\tau_1\tau_2}}{2\tau_1\tau_2}$$

$$p_1, p_2 = \frac{-\left[\tau_1(1+K_2)+\tau_2\right] \pm \sqrt{\tau_1^2 - 2\tau_1\tau_2 + \tau_2^2 + \left(\tau_1^2K_2^2 + 2\tau_1^2K_2 + 2K_2\tau_1\tau_2\right)}}{2\tau_1\tau_2}$$

$$p_1, p_2 = \frac{-\left[\tau_1(1+K_2)+\tau_2\right] \pm \sqrt{(\tau_1-\tau_2)^2 + \left(\tau_1^2K_2^2 + 2\tau_1^2K_2 + 2K_2\tau_1\tau_2\right)}}{2\tau_1\tau_2}$$

Notice that the resulting system is still 2nd order, stable, and overdamped. As an example, what happens to the poles when the tanks are identical? Here, $\tau_1 = \tau_2 = \tau$ and $K_2 = 1$. Then:

$$p_1, p_2 = \frac{-\left[\tau(1+1)+\tau\right] \pm \sqrt{(\tau-\tau)^2 + \left(\tau^2 + 2\tau^2 + 2\tau^2\right)}}{2\tau^2}$$

$$p_1, p_2 = \frac{-3\tau \pm \sqrt{5\tau^2}}{2\tau^2} = \frac{1}{\tau}\left(\frac{-3 \pm \sqrt{5}}{2}\right) = -\frac{0.382}{\tau}, -\frac{2.618}{\tau}$$

N Identical Processes in Series

When all of the processes in series are identical then the overall transfer function will become:

$$\frac{\bar{y}_N(s)}{\bar{f}(s)} = \prod_{i=1}^{N} G(s) = G^N .$$

This could lead to a very simple solution depending upon the forcing function. For example, if the processes are all first order and the forcing function is an impulse, $f = C \cdot \delta(t) \Rightarrow \bar{f} = C$, then the output response will be:

$$\begin{aligned}\bar{y}_N &= \left(\frac{K}{\tau s+1}\right)^N \cdot C \\ &= \left(\frac{K}{\tau}\right)^N \frac{1}{(N-1)!} \frac{(N-1)!}{\left(s+\frac{1}{\tau}\right)^N} \cdot C\end{aligned}$$

$$y_N(t) = \left(\frac{K}{\tau}\right)^N \frac{t^{N-1} e^{-t/\tau}}{(N-1)!} \cdot C .$$

The following figure shows the response curves for an impulse disturbance to a series of identical 1st order processes (with $C = 1$, $K = 1$ & $\tau = 2$). Notice that the response from the 1st process is an exponential & the responses from the rest of the processes look like spread pulses. The "peak" or maximum value of the disturbance occurs later for each additional process & (in this case) the peak is a little smaller for each additional process. In fact, we can determine when the peak will occur – it will be at that time t_m where $t_m > 0$ and the first time derivative of the response curve will be zero:

$$\frac{dy_N}{dt} = \frac{C}{(N-1)!} \left(\frac{K}{\tau}\right)^N \left[(N-1) t^{N-2} e^{-t/\tau} - \frac{t^{N-1} e^{-t/\tau}}{\tau} \right]$$

$$\frac{C}{(N-1)!} \left(\frac{K}{\tau}\right)^N \left[(N-1) t_m^{N-2} e^{-t_m/\tau} - \frac{t_m^{N-1} e^{-t_m/\tau}}{\tau} \right] = 0$$

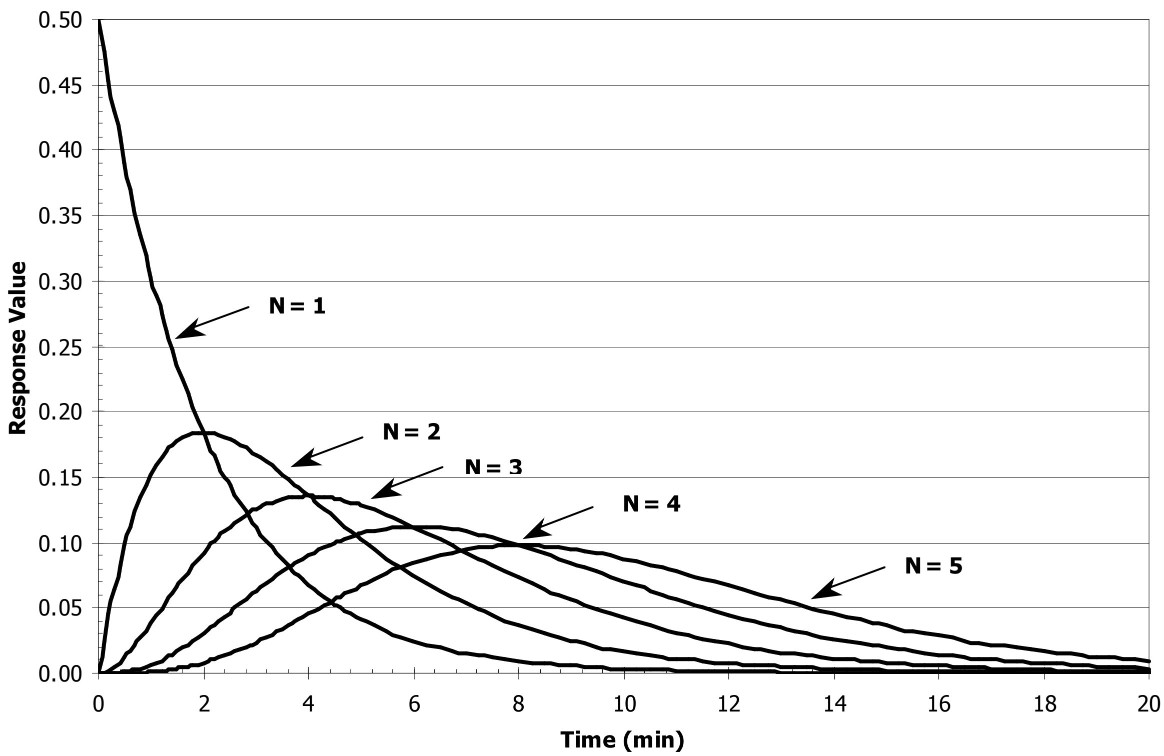
$$\frac{C}{(N-1)!} \left(\frac{K}{\tau}\right)^N \left[(N-1) - \frac{t_m}{\tau} \right] t_m^{N-2} e^{-t_m/\tau} = 0$$

$$(N-1) - \frac{t_m}{\tau} = 0$$

$$t_m = (N-1)\tau .$$

The corresponding value of the peak will be:

$$y_N(t_m) = C\left(\frac{K}{\tau}\right)^N \frac{\left[(N-1)\tau\right]^{N-1} e^{-(N-1)}}{(N-1)!} = \frac{CK^N}{\tau}\frac{(N-1)^{N-1} e^{-(N-1)}}{(N-1)!} .$$



Impulse Disturbance to Identical 1st Order Processes in Series

A more complicated forcing function will lead to more complicated Laplace functions that need to be inverted. However, with some ingenuity, it may be fairly easy to work with these more complicated functions. For example, if the forcing function is a step function, $f = A\cdot S(t) \Rightarrow \overline{f} = A/s$, then:

$$\overline{y}_N = \left(\frac{K}{\tau s+1}\right)^N \cdot \frac{A}{s}$$

and the partial fraction expansion must have all of the powers of the $\tau s+1$ term up to the $N-1$ power:

$$\overline{y}_N = AK^N\left[\frac{C_0}{s}+\sum_{i=1}^{N}\frac{C_i}{(\tau s+1)^i}\right] \;\Rightarrow\; y_N = AK^N\left[C_0 + e^{-t/\tau}\sum_{i=1}^{N}\frac{C_i}{\tau^i(i-1)!}\cdot t^{i-1}\right]$$

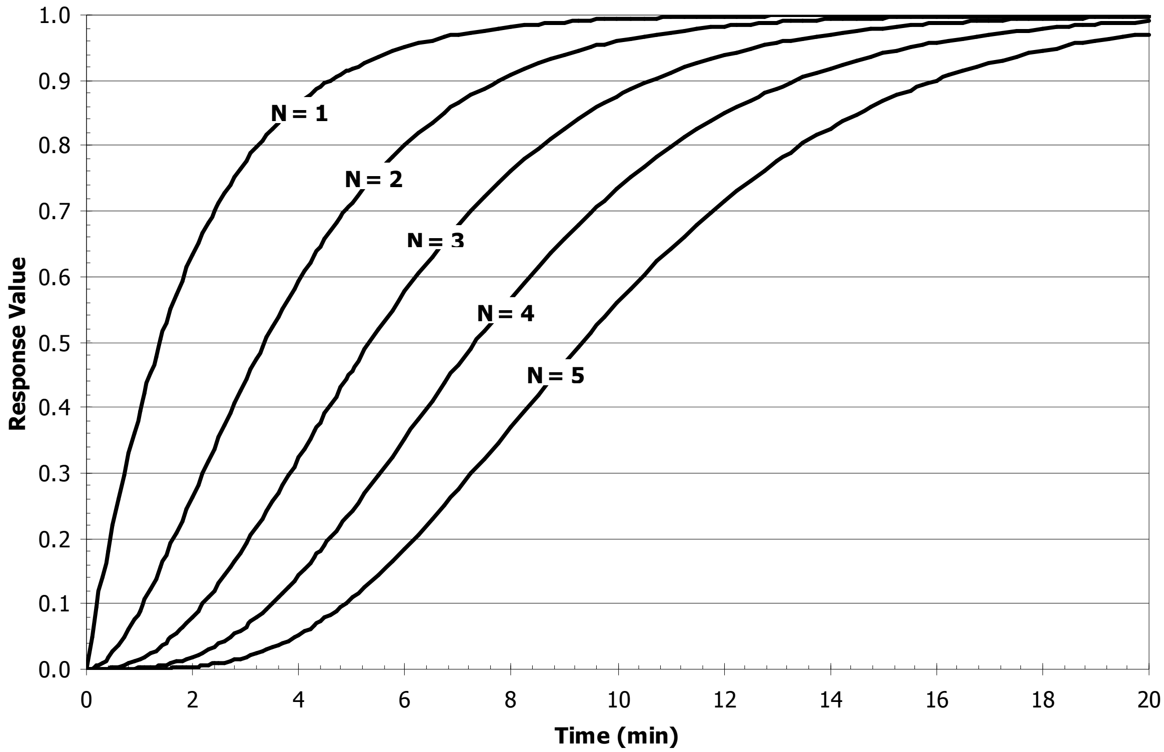
where the C_i coefficients must be determined to be consistent with the original overall transfer function. One's first thought is that these coefficients will be very difficult & tedious to determine. On the contrary, using the Heaviside method we can find:

$$C_i = \begin{cases} 1 & i=0 \\ -\tau & i=1,2,\ldots,N \end{cases}$$

so:

$$\overline{y}_N = AK^N\left[\frac{1}{s}-\sum_{i=1}^{N}\frac{\tau}{(\tau s+1)^i}\right] \;\Rightarrow\; y_N = AK^N\left[1 - e^{-t/\tau}\sum_{i=1}^{N}\frac{1}{\tau^{i-1}(i-1)!}\cdot t^{i-1}\right].$$

The following figure shows the response curves for a step disturbance to a series of identical 1st order processes (with $A=1$, $K=1$ & $\tau=2$). Notice that the higher order response curves look very much like the 2nd order response (i.e., that the initial response has a zero slope) but the time period in which there is very little response is much longer.



Step Disturbance to Identical 1st Order Processes in Series

Calculation of the Step Disturbance Coefficients

The transfer function to be split apart should be put into the following working form:

$$\frac{1}{s}\cdot\frac{1}{\left(\tau s+1\right)^{N}}=\frac{C_0}{s}+\sum_{i=1}^{N}\frac{B_i}{\left(s+\dfrac{1}{\tau}\right)^{i}}$$

where the B_i coefficients are related to the C_i coefficients by:

$$C_i = B_i \cdot \tau^i .$$

The first coefficient, C_0, is very easily calculated with the Heaviside method:

$$C_0=\lim_{s\to 0}\left[s\cdot\frac{1}{s}\cdot\frac{1}{\left(\tau s+1\right)^{N}}\right]=1 .$$

The coefficient for the highest order term is also very easily calculated:

$$B_N = \lim_{s\to -1/\tau}\left[\left(s+\frac{1}{\tau}\right)^N \cdot \frac{1}{s}\cdot\frac{1}{(\tau s+1)^N}\right] = \lim_{s\to -1/\tau}\left[\frac{1}{\tau^N}\cdot\frac{1}{s}\right] = -\frac{\tau}{\tau^N} \;\Rightarrow\; C_N = -\tau .$$

The other coefficients are calculated using the formula:

$$B_{N+1-k} = \lim_{s\to -1/\tau}\left[\frac{1}{(k-1)!}\cdot\frac{d^{k-1}}{ds^{k-1}}\left[\frac{\left(s+\frac{1}{\tau}\right)^N}{s(\tau s+1)^N}\right]\right] \quad k = 2,3,\ldots,N$$

$$B_{N+1-k} = \lim_{s\to -1/\tau}\left[\frac{1}{(k-1)!}\cdot\frac{1}{\tau^N}\cdot\frac{d^{k-1}}{ds^{k-1}}\left[\frac{1}{s}\right]\right]$$

The derivatives of $1/s$ have a fairly simple form:

$$\frac{d^n}{ds^n}\left[\frac{1}{s}\right] = (-1)^n \cdot n! \cdot \frac{1}{s^{n+1}}$$

So:

$$\begin{aligned} B_{N+1-k} &= \lim_{s\to -1/\tau}\left[\frac{1}{(k-1)!}\cdot\frac{1}{\tau^N}\cdot(-1)^{(k-1)}\cdot(k-1)!\cdot\frac{1}{s^k}\right] \\ &= \lim_{s\to -1/\tau}\left[\frac{(-1)^{(k-1)}}{\tau^N s^k}\right] \\ &= (-1)^{(k-1)}(-\tau)^k(\tau)^{-N} \\ &= (-1)^{(k-1)}(-1)^k\,\tau^{k-N} \\ &= (-1)^{(2k-1)}\,\tau^{k-N} \\ &= -\tau^{k-N} \end{aligned}$$

Note that since $2k-1$ will always be odd then $(-1)^{(2k-1)}$ will always be -1. Finally:

$$C_{N+1-k} = B_{N+1-k}\cdot\tau^{N+1-k} = -\tau^{k-N}\cdot\tau^{N+1-k} = -\tau .$$

So this shows that:

$$\frac{1}{s(\tau s+1)^N} = \frac{1}{s} - \sum_{i=1}^{N} \frac{\tau}{(\tau s+1)^i}.$$

Processes With Dead Time

So far we have assumed that whenever a change occurs, it reaches the measurement device instantaneously. However, there can be dead time between the process & the measurement device and we get transportation lag. One can picture the situation as being similar to plug flow through a pipe — the material does not change in the pipe, but there is a time difference between when it enters and when it reappears at the other end.

Since:

$$y(t+t_d) = f(t) \quad \Rightarrow \quad y(t) = f(t-t_d)$$

then the transfer function is immediately:

$$\bar{y}(s) = e^{-t_d s}\bar{f}(s)$$

We can develop this Laplace formula be thinking of dead time as an infinite number of identical non-interacting processes in series. Image a series of N tanks, each with the same volume, with a total volume of V. Into the first tank we make a change to the composition of component A. The component balance around the first tank will be:

$$V_1 \frac{dC'_{A1}}{dt} = F_0 C'_{A0} - F_0 C'_{A1}$$

and the corresponding transfer function will be:

$$\frac{\bar{C}'_{A1}}{\bar{C}'_{A0}} = \frac{1}{\left(\frac{V_1}{F_0}\right)s+1} = \frac{1}{\left(\frac{V}{NF_0}\right)s+1}.$$

Out of the Nth process:

$$\frac{\bar{C}'_{A,N}}{\bar{C}'_{A,0}} = \left(\frac{1}{\frac{V}{NF_0}s+1}\right)^N \quad \Rightarrow \quad \frac{\bar{C}'_{A,0}}{\bar{C}'_{A,N}} = \left(1+\frac{Vs}{NF_0}\right)^N = \left(1+\frac{t_d s}{N}\right)^N$$

where $t_d \equiv V / F_0$. It is known from advanced calculus that:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r} = e \quad \Rightarrow \quad \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n} = e^r$$

so as $N \to \infty$ then:

$$\frac{\bar{C}'_{\text{A},0}}{\bar{C}'_{\text{A},N}} = \left(1+\frac{t_d s}{N}\right)^N \to \exp(t_d s)$$

and:

$$G(s) = \frac{\bar{C}'_{\text{A},N}}{\bar{C}'_{\text{A},0}} \to \exp(-t_d s).$$

It is not always convenient (or even possible) to work with the dead-time term as it is defined. Remember that to invert the Laplace functions in the presence of a time delay term, all of the time delay terms must be factored out of & away from the rest of the Laplace expression. For example, the time delay term in the following expression can be factored out & the remain expression inverted:

$$\bar{y}' = \frac{Ke^{-\theta s}}{\tau s+1} = \frac{K}{\tau s+1}\cdot e^{-\theta s} \quad \Rightarrow \quad \begin{aligned} \mathcal{L}^{-1}\left[\frac{K}{\tau s+1}\right] &= \frac{K}{\tau}\exp\left(-\frac{t}{\tau}\right) \\ \mathcal{L}^{-1}\left[\frac{K}{\tau s+1}\cdot e^{-\theta s}\right] &= \frac{K}{\tau}\exp\left(-\frac{t-\theta}{\tau}\right)\cdot S(t-\theta) \end{aligned}.$$

However, the following expression comes about when we consider systems with a feedback of information:

$$\bar{y}' = \frac{\dfrac{Ke^{-\theta s}}{\tau s+1}}{1+K_c\dfrac{Ke^{-\theta s}}{\tau s+1}} = \frac{K}{\tau s+1+K_c K e^{-\theta s}}\cdot e^{-\theta s}.$$

We've factored the $e^{-\theta s}$ time delay term out of the numerator but it is still in the denominator & it cannot be removed form here. We cannot use the standard methods to invert this Laplace function.

When we have a situation like that above we will often approximate the exponential time delay term with a ratio of polynomials. Once we do this the exponential term is gone & we can use standard methods to break apart the transfer function into partial fractions and then invert. Approximate methods for dealing with time delays are based upon approximating the exponential term with some type of polynomial. Remember that the Taylor series expansion formulas is:

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=x_0} (x - x_0)^n$$

so the exponential function expanded around $x = 0$ is:

$$e^{+\theta s} = \sum_{i=0}^{\infty} \frac{\theta^n}{n!} x^n = 1 + \theta x + \frac{(\theta x)^2}{2} + \frac{(\theta x)^3}{6} + \cdots$$

and for the actual time delay term:

$$e^{-\theta s} = \sum_{i=0}^{\infty} \left({}^{n}-1\right) \frac{\theta^n}{n!} x^n = 1 - \theta x + \frac{(\theta x)^2}{2} - \frac{(\theta x)^3}{6} + \cdots$$

One could just truncate the polynomial after a prescribed number of terms. So we could use linear forms of either:

$$e^{-\theta s} \approx 1 - \theta x$$

or:

$$e^{-\theta s} = \frac{1}{e^{+\theta s}} \approx \frac{1}{1 + \theta x}.$$

And, depending upon how the approximation will be used, this may be perfectly acceptable. However, it is more typical that the exponential term is approximated with a <u>ratio of polynomials</u>. The most typical set of functions used for the time delay term are the Padé approximations. There are approximations of various order. The most common one used (especially for hand calculations) is the 1st order Padé approximation:

$$e^{-\theta s} \approx \frac{1 - \frac{1}{2}\theta s}{1 + \frac{1}{2}\theta s} = \frac{2 - \theta s}{2 + \theta s}$$

Higher order approximations are:

$$e^{-\theta s} \approx \frac{1-\frac{1}{2}(\theta s)+\frac{1}{12}(\theta s)^2}{1+\frac{1}{2}(\theta s)+\frac{1}{12}(\theta s)^2} = \frac{12-6(\theta s)+(\theta s)^2}{12+6(\theta s)+(\theta s)^2}$$

$$e^{-\theta s} \approx \frac{1-\frac{1}{2}(\theta s)+\frac{1}{10}(\theta s)^2-\frac{1}{120}(\theta s)^3}{1+\frac{1}{2}(\theta s)+\frac{1}{10}(\theta s)^2+\frac{1}{120}(\theta s)^3} = \frac{120-60(\theta s)+12(\theta s)^2-(\theta s)^3}{120+60(\theta s)+12(\theta s)^2+(\theta s)^3}$$

and so on.

Padé approximations match terms to the exponential function's Taylor series expansion one order higher than the functions used in the Padé approximation. The 1st order approximation has a Taylor series expansion:

$$e^{-\theta s} \approx \frac{1-\frac{1}{2}\theta s}{1+\frac{1}{2}\theta s} = 1-(\theta s)+\frac{(\theta s)^2}{2}-\frac{(\theta s)^3}{4}+\cdots$$

and so it actually matches terms up to the 2nd power, the 2nd order approximation has a Taylor series expansion:

$$e^{-\theta s} \approx \frac{1-\frac{1}{2}(\theta s)+\frac{1}{12}(\theta s)^2}{1+\frac{1}{2}(\theta s)+\frac{1}{12}(\theta s)^2} == 1-(\theta s)+\frac{(\theta s)^2}{2}-\frac{(\theta s)^3}{6}+\frac{(\theta s)^4}{18}-\cdots$$

and matches terms up to the 3rd power, and so forth. You should get higher order accuracy using lower order polynomials.

The response curves using Padé approximations tend to be least accurate at short times (especially on the order of the time delay θ) and become more accurate with higher order approximations. However, the difficulty of working with the expressions becomes much greater with higher order approximations. For example, using a 1st order approximation, the example transfer function is approximated as:

$$\overline{y}' = \frac{Ke^{-\theta s}}{\tau s+1+K_c K e^{-\theta s}} \approx \frac{K\frac{2-\theta s}{2+\theta s}}{\tau s+1+K_c K\frac{2-\theta s}{2+\theta s}} = \frac{K(2-\theta s)}{(2+\theta s)(\tau s+1)+K_c K(2-\theta s)}.$$

This is a transfer function for a 2nd order process and the inverse can be determined using standard forms. Using a 2nd order approximation, the example transfer function is:

$$\bar{y}' = \frac{Ke^{-\theta s}}{\tau s + 1 + K_c K e^{-\theta s}} \approx \frac{K \cdot \dfrac{12 - 6(\theta s) + (\theta s)^2}{12 + 6(\theta s) + (\theta s)^2}}{\tau s + 1 + K_c K \cdot \dfrac{12 - 6(\theta s) + (\theta s)^2}{12 + 6(\theta s) + (\theta s)^2}}$$
$$= \frac{K\left[12 - 6(\theta s) + (\theta s)^2\right]}{\left[12 + 6(\theta s) + (\theta s)^2\right](\tau s + 1) + K_c K\left[12 - 6(\theta s) + (\theta s)^2\right]} .$$

This is a transfer function for a 3rd order process and it must be split apart using partial fractions into the sum of a 1st order response & a 2nd order response before standard forms can be used to determine the inverse.

Approximate Reduced Order Transfer Functions – FOPDT

Even though we've spent a lot of time developing transfer functions of high order, it may be convenient to work with an approximate low order transfer function that will give essentially the same type of dynamic response. The slow response at small times (due to the smallest time constants) can be approximated with a time delay term. Keeping only one time constant gives rise to an FOPDT transfer function (First Order Process with Time Delay).

One way to reduce the order of a transfer function is approximate the small time constants with corresponding time delay terms. For example, if $\tau_1 > \tau_2 > \tau_3 > \tau_4$ is the largest time constant, the following 4th order transfer function can be approximated with the following FOPDT model:

$$\frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)(\tau_4 s + 1)} = \frac{K_p}{\tau_1 s + 1} \cdot \frac{1}{\tau_2 s + 1} \cdot \frac{1}{\tau_3 s + 1} \cdot \frac{1}{\tau_4 s + 1}$$
$$\approx \frac{K_p}{\tau_1 s + 1} \cdot \frac{1}{\tau_2 s + 1} \cdot \frac{1}{\tau_3 s + 1} \cdot \frac{1}{\tau_4 s + 1}$$
$$= \frac{K_p}{\tau_1 s + 1} \cdot e^{-\tau_2 s} \cdot e^{-\tau_3 s} \cdot e^{-\tau_4 s}$$
$$= \frac{K_p e^{-(\tau_2 + \tau_3 + \tau_4)s}}{\tau_1 s + 1}$$

An approximate SOPDT model (Second Order Process with Dead Time) would be:

$$\frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)(\tau_4 s+1)} \approx \frac{K_p e^{-(\tau_3+\tau_4)s}}{(\tau_1 s+1)(\tau_2 s+1)}.$$

Skogestad has proposed a related method. He suggests that the largest neglected time constant should be split between the smallest retained time constant and the time delay. So, Skogestad's FOPDT model would be:

$$\frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)(\tau_4 s+1)} \approx \frac{K_p e^{-(\tau_2/2+\tau_3+\tau_4)s}}{(\tau_1+\tau_2/2)s+1}$$

and the SOPDT model would be:

$$\frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)(\tau_4 s+1)} \approx \frac{K_p e^{-(\tau_3/2+\tau_4)s}}{(\tau_1 s+1)\left[(\tau_2+\tau_3/2)s+1\right]}.$$

These methods are best applied when you already have the coefficients for the high order transfer functions. If you must first derive the coefficients of the high order model from empirical data, it is better to derive the coefficients of the FOPDT or SOPDT model directly.

FOPDT Example

The 4th order transfer function:

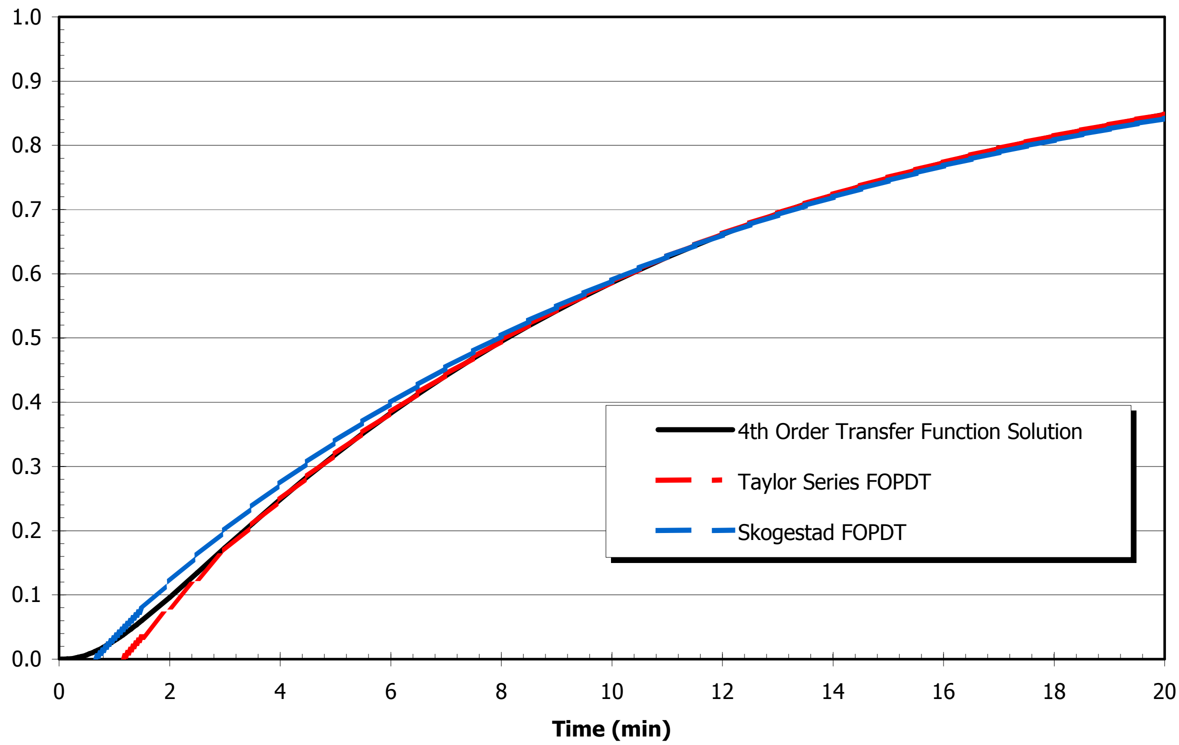
$$G(s) = \frac{1}{(10s+1)(s+1)(0.1s+1)(0.05s+1)}$$

can be approximated as the following FOPDT model using 1st order Taylor series expansions:

$$G(s) \approx \frac{e^{-1.15s}}{10s+1}$$

or the following FOPDT model using the Skogestad method:

$$G(s) \approx \frac{e^{-0.65s}}{10.5s+1}.$$



The figure above compares the response curves for a step change forcing function for the original transfer function & the two FOPDT models. Notice that both FOPDT models match the response from the original transfer function once past $t \sim 8\text{ min}$.

Processes With Inverse Response

When the overall transfer function is found as the difference of individual transfer functions, then the initial response in a system may be in the opposite direction than where it will end up. One example is adding cold feedwater to a drum reboiler to control the liquid level. The 1st bit of cold feedwater will decrease the temperature, causing the volume of the entrained vapor to decrease. The may be the overwhelming effect at first, causing the overall liquid level to drop. The liquid level will then increase again once the heat is added.

The overall transfer function for a process that shows inverse response will look something like:

$$G(s) = G_1(s) - G_2(s)$$

where $G_2(s)$ dominates at early times and $G_1(s)$ dominates at long times. Expressing the overall transfer function as a ratio of two polynomials:

$$G(s)=\frac{N_m(s)}{D_n(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

then there will be an inverse response when there is at least one positive zero in the transfer function (i.e., there is at least one positive root to the polynomial in the numerator $N_m(s)$).
Let's look at the necessary conditions for inverse response when combining several types of common transfer functions.

Inverse Response from the Sum of Two 1st Order Transfer Functions

The transfer function formed from the sum of two 1st order transfer functions is:

$$G(s)=\frac{K_1}{\tau_1 s+1}+\frac{K_2}{\tau_2 s+1}=\frac{(K_1\tau_2+K_2\tau_1)s+(K_1+K_2)}{(\tau_1 s+1)(\tau_2 s+1)}$$

and the root of the numerator will be:

$$s=-\frac{K_1+K_2}{K_1\tau_2+K_2\tau_1} \text{ implies an inverse response for } \frac{K_1+K_2}{K_1\tau_2+K_2\tau_1}<0.$$

If we only consider positive gains & time constants then there will not be an inverse response — the combination of positive terms cannot give a negative number.

Inverse Response from the Difference Between Two 1st Order Transfer Functions

The transfer function formed from the difference of two 1st order transfer functions is:

$$G(s)=\frac{K_1}{\tau_1 s+1}-\frac{K_2}{\tau_2 s+1}=\frac{(K_1\tau_2-K_2\tau_1)s+(K_1-K_2)}{(\tau_1 s+1)(\tau_2 s+1)}$$

and the root of the numerator will be:

$$s=-\frac{K_1-K_2}{K_1\tau_2-K_2\tau_1} \text{ implies an inverse response for } \frac{K_1-K_2}{K_1\tau_2-K_2\tau_1}<0.$$

Inverse response will occur if the numerator & denominator are of opposite sign so that overall the fraction is negative. So, there are two possibilities. If:

$$K_1 - K_2 > 0 \quad \Rightarrow \quad K_1 > K_2 \quad \Rightarrow \quad \frac{K_1}{K_2} > 1$$

then:

$$K_1\tau_2 - K_2\tau_1 < 0 \quad \Rightarrow \quad K_2\tau_1 > K_1\tau_2 \quad \Rightarrow \quad \frac{\tau_1}{\tau_2} > \frac{K_1}{K_2} > 1 .$$

However, we can also have:

$$K_1 - K_2 < 0 \quad \Rightarrow \quad K_1 < K_2 \quad \Rightarrow \quad \frac{K_1}{K_2} < 1$$

then:

$$K_1\tau_2 - K_2\tau_1 > 0 \quad \Rightarrow \quad K_2\tau_1 < K_1\tau_2 \quad \Rightarrow \quad \frac{\tau_1}{\tau_2} < \frac{K_1}{K_2} < 1 .$$

This shows that just having the difference of two 1st order transfer functions is not a sufficient condition for an inverse response.

We can also look at many more combinations of transfer functions to determine when we might get an inverse response.

Inverse Response from the Difference Between a 2nd & 1st Order Transfer Functions

Now, let's look at when the overall transfer function is the difference between 2nd order & 1st order transfer functions. Then:

$$G(s) = \frac{K_1}{\tau_1^2 s^2 + 2\zeta_1\tau_1 s + 1} - \frac{K_2}{\tau_2 s + 1} = \frac{\left(-K_2\tau_1^2\right)s^2 + \left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)s + \left(K_1 - K_2\right)}{\left(\tau_2 s + 1\right)\left(\tau_1^2 s^2 + 2\zeta_1\tau_1 s + 1\right)}$$

The roots to the numerator are:

$$r_1, r_2 = \frac{-\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right) \pm \sqrt{\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 - 4\left(-K_2\tau_1^2\right)\left(K_1 - K_2\right)}}{2\left(-K_2\tau_1^2\right)}$$

$$r_1, r_2 = \frac{K_1\tau_2 - 2K_2\zeta_1\tau_1 \pm \sqrt{\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 + 4K_2\tau_1^2\left(K_1 - K_2\right)}}{2K_2\tau_1^2}$$

The largest root will be with the positive sign, so to guarantee an inverse response we only need to restrict this root:

$$K_1\tau_2 - 2K_2\zeta_1\tau_1 + \sqrt{\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 + 4K_2\tau_1^2\left(K_1 - K_2\right)} > 0$$

$$\sqrt{\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 + 4K_2\tau_1^2\left(K_1 - K_2\right)} > -\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)$$

Note that the 1st term under the radical is:

$$\sqrt{\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2} > \left|K_1\tau_2 - 2K_2\zeta_1\tau_1\right|$$

If $-\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right) > 0$, then we only need the 2nd term under the radical to be positive for the inequality to hold:

$$\begin{aligned}
&4K_2\tau_1^2\left(K_1 - K_2\right) > 0 \\
&K_1 - K_2 > 0 \\
&K_1 > K_2 \\
&\frac{K_1}{K_2} > 1
\end{aligned}$$

which is what is given in the text. However, we really need to combine this with the 1st assumption to get:

$$\begin{aligned}
&-\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right) > 0 \\
&2K_2\zeta_1\tau_1 > K_1\tau_2 \\
&\frac{2\zeta_1\tau_1}{\tau_2} > \frac{K_1}{K_2}
\end{aligned}$$

and: $$\frac{2\zeta_1\tau_1}{\tau_2} > \frac{K_1}{K_2} > 1$$

But what if $-\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right) < 0$? Now the 2nd term under the radical can be negative and the inequality may still hold. Let's multiply the inequality by itself — the direction of the inequality must change since we are essentially multiplying it by a negative value. Then:

$$\begin{aligned}
&\left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 + 4K_2\tau_1^2\left(K_1 - K_2\right) < \left(K_1\tau_2 - 2K_2\zeta_1\tau_1\right)^2 \\
&K_1 - K_2 < 0
\end{aligned}$$

$$K_1 < K_2$$
$$\frac{K_1}{K_2} < 1.$$

Now when we combine this with the 1st assumption we get:

$$-(K_1\tau_2 - 2K_2\zeta_1\tau_1) < 0$$
$$2K_2\zeta_1\tau_1 < K_1\tau_2$$
$$\frac{2\zeta_1\tau_1}{\tau_2} < \frac{K_1}{K_2}$$

and: $$\frac{2\zeta_1\tau_1}{\tau_2} < \frac{K_1}{K_2} < 1$$

This allows for the possibility that the ultimate response is controlled by the 2nd (negative) transfer function, not the 1st (positive) one.

Inverse Response from the Difference Between Two 2nd Order Transfer Functions

Finally, let's look at the case when both of the individual transfer functions are 2nd order. Then:

$$\begin{aligned} G(s) &= \frac{K_1}{\tau_1^2 s^2 + 2\zeta_1\tau_1 s + 1} - \frac{K_2}{\tau_2^2 s^2 + 2\zeta_2\tau_2 s + 1} \\ &= \frac{(K_1\tau_2^2 - K_2\tau_1^2)s^2 + 2(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1)s + (K_1 - K_2)}{(\tau_1^2 s^2 + 2\zeta_1\tau_1 s + 1)(\tau_2^2 s^2 + 2\zeta_2\tau_2 s + 1)} \end{aligned}$$

Again there are two zeros to the 2nd order numerator given by the quadratic formula:

$$r_1, r_2 = \frac{-2(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1) \pm \sqrt{4(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1)^2 - 4(K_1\tau_2^2 - K_2\tau_1^2)(K_1 - K_2)}}{2(K_1\tau_2^2 - K_2\tau_1^2)}$$
$$r_1, r_2 = \frac{-(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1) \pm \sqrt{(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1)^2 - (K_1\tau_2^2 - K_2\tau_1^2)(K_1 - K_2)}}{K_1\tau_2^2 - K_2\tau_1^2}.$$

When the denominator is positive (i.e., $K_1\tau_2^2 - K_2\tau_1^2 > 0 \Rightarrow K_1\tau_2^2 > K_2\tau_1^2$) then the largest root is given by the "plus" sign on the radical — that is all that is necessary for an inverse response. Then:

$$-\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) + \sqrt{\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right)} > 0\,.$$
$$\sqrt{\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right)} > \left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right).$$

As we did for the previous set of transfer functions, we will multiply the inequality by itself. What happens to the inequality sign depends upon the sign of what is on the right hand side (since the radical on the left hand side has to be positive). If $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) > 0$ then:

$$\begin{aligned}
&\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) > \left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 \\
&-\left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) > 0 \\
&-\left(K_1 - K_2\right) > 0 \quad \text{since we've already assumed } K_1\tau_2^2 - K_2\tau_1^2 > 0 \\
&K_1 < K_2 \\
&\frac{K_1}{K_2} < 1
\end{aligned}$$

Now taking all of the conditions into account:

$$K_1\tau_2^2 - K_2\tau_1^2 > 0 \quad \Rightarrow \quad 1 > \frac{K_1}{K_2} > \frac{\tau_1^2}{\tau_2^2}$$

and: $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) > 0 \quad \Rightarrow \quad 1 > \dfrac{K_1}{K_2} > \dfrac{\zeta_1\tau_1}{\zeta_2\tau_2}$

However if $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) < 0$ then we must flip the inequality:

$$\begin{aligned}
&\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) < \left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 \\
&-\left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) < 0 \\
&-\left(K_1 - K_2\right) < 0 \quad \text{since we've already assumed } K_1\tau_2^2 - K_2\tau_1^2 > 0 \\
&K_1 > K_2 \\
&\frac{K_1}{K_2} > 1
\end{aligned}$$

Now when taking the other conditions into account we cannot compare them to 1:

$$K_1\tau_2^2 - K_2\tau_1^2 > 0 \quad \Rightarrow \quad \frac{K_1}{K_2} > \frac{\tau_1^2}{\tau_2^2}$$

and: $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) > 0 \quad \Rightarrow \quad \dfrac{K_1}{K_2} > \dfrac{\zeta_1\tau_1}{\zeta_2\tau_2}$

We still have another pair of cases to consider, that when the denominator is negative (i.e., $K_1\tau_2^2 - K_2\tau_1^2 < 0$). Now, the largest root is given by the "minus" sign on the radical. Then:

$$-\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) - \sqrt{\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right)} < 0 .$$
$$\sqrt{\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right)} > -\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right).$$

Again, what happens to the inequality sign depends upon the sign of what is on the right hand side. If $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) > 0$ then we must flip the inequality:

$$\begin{aligned}
&\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) < \left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 \\
&-\left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) < 0 \\
&-\left(K_1 - K_2\right) > 0 \quad \text{since we've already assumed } K_1\tau_2^2 - K_2\tau_1^2 < 0 \\
&K_1 < K_2 \\
&\frac{K_1}{K_2} < 1
\end{aligned}$$

Now taking all of the conditions into account:

$K_1\tau_2^2 - K_2\tau_1^2 < 0 \quad \Rightarrow \quad \dfrac{K_1}{K_2} < \dfrac{\tau_1^2}{\tau_2^2}$

and: $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) < 0 \quad \Rightarrow \quad \dfrac{K_1}{K_2} < \dfrac{\zeta_1\tau_1}{\zeta_2\tau_2}.$

Finally, if $\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right) < 0$ then we keep the inequality the same:

$$\begin{aligned}
&\left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 - \left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) > \left(K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1\right)^2 \\
&-\left(K_1\tau_2^2 - K_2\tau_1^2\right)\left(K_1 - K_2\right) > 0 \\
&-\left(K_1 - K_2\right) < 0 \quad \text{since we've already assumed } K_1\tau_2^2 - K_2\tau_1^2 < 0 \\
&K_1 > K_2 \\
&\frac{K_1}{K_2} > 1
\end{aligned}$$

Now when taking the other conditions into account we cannot compare them to 1:

$$K_1\tau_2^2 - K_2\tau_1^2 < 0 \quad \Rightarrow \quad 1 < \frac{K_1}{K_2} < \frac{\tau_1^2}{\tau_2^2}$$

and: $$K_1\zeta_2\tau_2 - K_2\zeta_1\tau_1 < 0 \quad \Rightarrow \quad 1 < \frac{K_1}{K_2} < \frac{\zeta_1\tau_1}{\zeta_2\tau_2}.$$