Ziegler-Nichols Controller Tuning Example

The Ziegler-Nichols method uses a closed controller loop & requires the following steps:

- Bring system to steady state operation.
- Put on P control. Introduce a set point change and vary gain until system oscillates continuously. This frequency is ω_{CO} and M is the amplitude ratio.
- Compute the following:

$$\text{Ultimate Gain} = K_u = \frac{1}{M}$$

$$\text{Ultimate Period} = P_u = \frac{2\pi}{\omega_{CO}}$$

The original Z-N tuning settings are given in the following table.

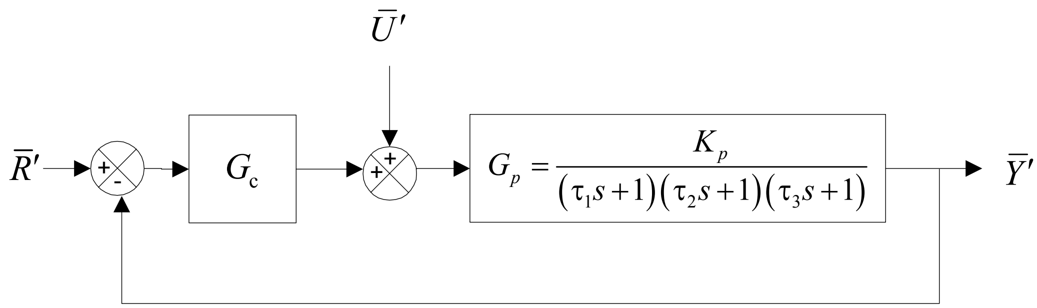
Controller	K_c	τ_I	τ_D
P	$K_u/2$	—	—
PI	$K_u/2.2$	$P_u/1.2$	—
PID	$K_u/1.7$	$P_u/2$	$P_u/8$

These controller settings were developed to give a 1/4 decay ratio. However, other settings have been recommended that are closer to critically damped control (so that oscillations do not propagate downstream). These PID controller settings are shown in the following table.

Controller	K_c	τ_I	τ_D
Original	$0.6K_u$	$P_u/2$	$P_u/8$
Some Overshoot	$0.33K_u$	$P_u/2$	$P_u/3$
No Overshoot	$0.22K_u$	$P_u/2$	$P_u/3$

The question becomes one of how to find these parameters when we don't have a real process but rather the idealized transfer functions for the process.

General 3rd Order Overdamped System Example



As an example, let's assume we have a 3rd order overdamped system with negligible dynamics in the final control & measurement elements. Then:

$$G_p(s) = \frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$
$$G_f(s) = G_m(s) = 1$$

The transfer function between the set point and the output under P control will be:

$$G(s) = \frac{G_c G_p}{1+G_c G_p} = \frac{K_c \frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}}{1+K_c \frac{K_p}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}}$$
$$= \frac{K_c K_p}{\tau_1\tau_2\tau_3 s^3 + (\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)s^2 + (\tau_1+\tau_2+\tau_3)s + (1+K_p K_c)}$$

The characteristic equation of this transfer function:

$$\tau_1\tau_2\tau_3 s^3 + (\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)s^2 + (\tau_1+\tau_2+\tau_3)s + (1+K_p K_c) = 0$$

will control the stability & oscillatory nature of the system's response. Undamped oscillatory behavior occurs at the very edge of the system's stability limit.

Routh Array. We can use a Routh array to determine this point.

Row		
1	$\tau_1\tau_2\tau_3$	$\tau_1+\tau_2+\tau_3$
2	$\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1$	$1+K_pK_c$
3	$\tau_1+\tau_2+\tau_3-\dfrac{\tau_1\tau_2\tau_3\left(1+K_pK_c\right)}{\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1}$	
4	$1+K_pK_c$	

The stability limit from the 4th row is:

$$1+K_pK_c>0 \quad\Rightarrow\quad K_c>-\frac{1}{K_p}$$

which will be satisfied for any positive K_c. The stability limit from the 3rd row is:

$$\tau_1+\tau_2+\tau_3-\frac{\tau_1\tau_2\tau_3\left(1+K_pK_c\right)}{\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1}>0 \quad\Rightarrow\quad K_c<\frac{\left(\tau_1+\tau_2+\tau_3\right)\left(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1\right)}{K_p\tau_1\tau_2\tau_3}-\frac{1}{K_p}$$

$$K_c<\frac{1}{K_p}\left(\tau_1+\tau_2+\tau_3\right)\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{1}{\tau_3}\right)-\frac{1}{K_p}$$

So, the ultimate gain will correspond to the stability limit:

$$K_{cu}=\frac{1}{K_p}\left(\tau_1+\tau_2+\tau_3\right)\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{1}{\tau_3}\right)-\frac{1}{K_p}$$

The period of oscillation at this ultimate gain must be calculated from the inverted Laplace expression.

Direct Substitution. We could also use direct substitution to determine the stability limit for this process. Substituting $s=\omega j$ into the characteristic equation:

$$\tau_1\tau_2\tau_3\left(\omega j\right)^3+\left(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1\right)\left(\omega j\right)^2+\left(\tau_1+\tau_2+\tau_3\right)\left(\omega j\right)+\left(1+K_pK_c\right)=0$$

$$-\tau_1\tau_2\tau_3\left(\omega^3 j\right)-\left(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1\right)\left(\omega^2\right)+\left(\tau_1+\tau_2+\tau_3\right)\left(\omega j\right)+\left(1+K_pK_c\right)=0$$

$$\left[\left(1+K_pK_c\right)-\left(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1\right)\omega^2\right]+\left[\left(\tau_1+\tau_2+\tau_3\right)\omega-\tau_1\tau_2\tau_3\omega^3\right]j+=0.$$

Setting the imaginary part to zero gives us the frequency of oscillation:

$$(\tau_1+\tau_2+\tau_3)\omega_u-\tau_1\tau_2\tau_3\omega_u^3=0 \quad\Rightarrow\quad \omega_u=\sqrt{\frac{\tau_1+\tau_2+\tau_3}{\tau_1\tau_2\tau_3}}$$

and setting the real part to zero gives us the controller gain at the stability limit:

$$(1+K_pK_{cu})-(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)\omega_u^2=0 \quad\Rightarrow K_{cu}=\frac{(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)\frac{\tau_1+\tau_2+\tau_3}{\tau_1\tau_2\tau_3}-1}{K_p}$$

$$=\frac{1}{K_p}\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}+\frac{1}{\tau_3}\right)(\tau_1+\tau_2+\tau_3)-\frac{1}{K_p}$$

(which is exactly the same as that given by the Routh array analysis).

Specific 3rd Order Overdamped System Example

Let us take for example the process:

$$G_p(s)=\frac{1}{(s+1)(5s+1)(0.2s+1)}$$

The ultimate gain will be:

$$K_{cu}=\frac{1}{1}(1+5+0.2)\left(\frac{1}{1}+\frac{1}{5}+\frac{1}{0.2}\right)-\frac{1}{1}=37.44$$

and the frequency of oscillation will be:

$$\omega_{CO}=\sqrt{\frac{1+5+0.2}{1\cdot 5\cdot 0.2}}=2.48998$$

so the period of oscillation at the ultimate gain is:

$$P_u=\frac{2\pi}{\omega_{CO}}=2.52339$$

The following will be the Ziegler-Nichols controller settings:

- P control:

$$K_c = \frac{K_u}{2} = 18.72$$

- PI control:

$$K_c = \frac{K_u}{2.2} = 17.02$$
$$\tau_I = \frac{P_u}{1.2} = 2.10$$

- PID control:

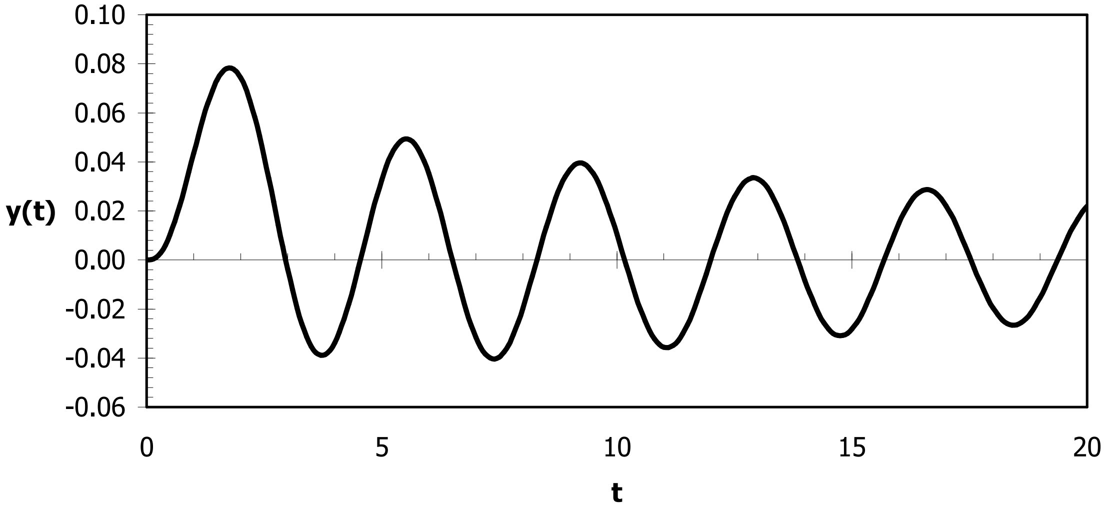
$$K_c = \frac{K_u}{1.7} = 22.02$$
$$\tau_I = \frac{P_u}{2} = 1.26$$
$$\tau_D = \frac{P_u}{8} = 0.32$$

Let's look at a step change to the load for PI control. The closed loop transfer function will be:

$$G(s) = \frac{G_p}{1 + G_c G_p} = \frac{\dfrac{1}{(s+1)(5s+1)(0.2s+1)}}{1 + K_c\left(1 + \dfrac{1}{\tau_I s}\right)\dfrac{1}{(s+1)(5s+1)(0.2s+1)}}$$
$$= \frac{\tau_I s}{\tau_I s(s+1)(5s+1)(0.2s+1) + K_c(\tau_I s + 1)}$$
$$= \frac{\tau_I s}{\tau_I s^4 + 6.2\tau_I s^3 + 6.2\tau_I s^2 + \tau_I(1 + K_c)s + K_c}$$

With a step change to the load, $\overline{U} = 1/s$ and the Ziegler-Nichols settings:

$$\overline{Y}(s) = \frac{1}{8.1039 + 18.0182s + 6.2s^2 + 6.2s^3 + s^4}$$
$$Y(t) = -0.00573006e^{-5.62165t} + 0.0626769e^{-0.495759t}$$
$$-e^{-0.0412961t}\left[0.00204812\sin(1.70472t) + 0.0569469\cos(1.70472t)\right]$$



This figure shows the response to this disturbance. The controller settings are very conservative – there is very little damping to the response. In actual practice, these settings would only be used as initial settings and additional tuning would be done.