Modeling the Quarter-Wave Transformer

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INTRODUCTION

The quarter-wave transformer is a section of transmission line, one quarter of a wavelength long, that is used to transform the impedance of a load (e.g., an antenna) to a new value that will match the characteristic impedance of a given transmission-line.[1] A model for the quarter-wave transformer has been described.[2] This paper examines an alternative model.[3]

In this paper, basics of the quarter-wave transformer and relevant transmission line theory are first presented. The discussion of theory includes reflection and transmission coefficients, and reflected and transmitted wave components. Then, the values of the voltage and current wave components on each section of the transformer are computed for an example system consisting of source, transformer, and load. These values are for specific points in time, starting from initialization of the system until the steady-state condition has been reached. Finally, given the steady-state values of the voltage and current wave components, the wave impedance and the power being delivered on each section of the transformer are calculated.

BASICS

The quarter-wave transformer with a load is depicted in Figure 1. A section of transmission line of characteristic impedance Z_S is fed by a source (not shown). The load is a section of transmission line of characteristic impedance Z_E terminated in a pure resistance $Z_A = R_A+j0$. These two sections of transmission line are joined by a third section of transmission line, the transformer, that is a quarter-wavelength long at the frequency of interest (the lengths of the other two sections do not matter). A requirement is that the characteristic impedance Z_C of the transmission line in the quarter-wave section must be

$$Z_C = (Z_S Z_E)^{1/2}$$

As seen in Figure 1, there are impedance discontinuities designated A and B at the two interfaces between the different lines. For a voltage or current wave approaching either of these discontinuities from any direction, two things occur according to standard transmission line theory: a portion of the incident wave is reflected at the discontinuity and sent back in the direction it came from, and a portion of the wave is transmitted and sent on in the same direction.

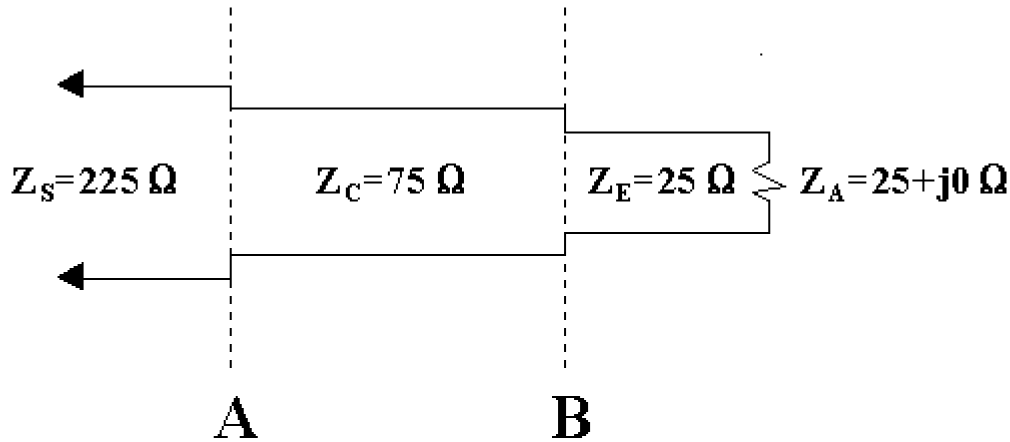

Figure 1 - The Quarter-Wave Transformer with Load

The Kraus and Carver model of the quarter-wave transformer [3] involves a series of partially reflected and partially transmitted wave components that occur over a period of time. Both voltage and current waves are considered. At the interfaces A and B, various individual wave components traveling in the same direction either cancel or reinforce each other, depending on their respective phase relationships. Eventually, steady-state conditions are achieved, at which time the magnitudes of the voltage and current waves are essentially constant. Because of the multitude of wave components occurring at various time instances, the best way to describe this model is by presenting a numerical example delineating the individual wave components at various time snap-shots prior to the steady-state condition. For purposes of comparison, the example used in [2] is analyzed using the method of [3].

The values for the quarter-wave transformer example from [2] are shown in Figure 1. Z_S is 225-ohm transmission line, and Z_E is 25-ohm transmission line. The characteristic impedance Z_C must then be 75 ohms. In this example, all transmission lines are assumed to be lossless.

Wave Components

For an incident voltage wave E_{inc} at an impedance discontinuity, the reflected voltage wave component E_{refl} is given by[4]

$$E_{refl} = \Delta_v E_{inc}$$

and the reflected current wave component I_{refl} for an incident current wave I_{inc} is given by

$$I_{refl} = \Delta_i I_{inc}$$

where Δ_v is the voltage reflection coefficient and Δ_i is the current reflection coefficient. Similarly, the transmitted voltage wave component E_{trans} at the discontinuity is given by

$$E_{trans} = \vartheta_v E_{inc}$$

and the transmitted current wave component is given by

$$I_{trans} = \vartheta_i I_{inc}$$

where ϑ_v is the voltage transmission coefficient and ϑ_i is the current transmission coefficient[5] .

Reflection Coefficients

In general, the voltage reflection coefficient Δ_v is given by:

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}$$

where Z_L is the load impedance for a section of transmission line of characteristic impedance Z_0. Note that Z_L can be another section of transmission line. The other coefficients, which are defined in terms of Δ_v, are:

$\Delta_i = -\Delta_v$	current reflection coefficient
$\vartheta_v = 1+\Delta_v$	voltage transmission coefficient
$\vartheta_i = 1+\Delta_i$	current transmission coefficient

For the Kraus and Carver model, three voltage reflection coefficients and three current reflection coefficients are germane. The reflection coefficients are as follows.

- Δ_v^{AS}, Δ_i^{AS} for wave components reflected back toward the source from A for a wave initially traveling in the direction of the load
- Δ_v^{AL}, Δ_i^{AL} for wave components reflected back toward the load from A for a wave initially traveling in the direction of the source
- Δ_v^{BS}, Δ_i^{BS} for wave components reflected back toward the source from B for a wave initially traveling in the direction of the load

The notation for each coefficient indicates the discontinuity and the direction of the reflected component. With respect to Figure 1, a “forward” wave is one traveling in the direction of the load (i.e., moving left-to-right), and a “backward” wave is traveling in the direction of the source, or right-to-left.

For the example transformer, the complex-valued voltage reflection coefficients in polar notation are as follows:

$$\rho_v^{AS} = \frac{75 - 225}{75 + 225} = -0.50 = 0.5 / 180$$

$$\rho_v^{AL} = \frac{225 - 75}{225 + 75} = +0.50 = 0.5 / 0$$

$$\rho_v^{BS} = \frac{25 - 75}{25 + 75} = -0.50 = 0.5 / 180$$

The complex-valued current reflection coefficients are:

$$\rho_i^{AS} = -\rho_v^{AS} = 0.5 / 0$$

$$\rho_i^{AL} = -\rho_v^{AL} = 0.5 / 180$$

$$\rho_i^{BS} = -\rho_v^{BS} = 0.5 / 0$$

Transmission Coefficients

There are three voltage transmission coefficients and three current transmission coefficients:

- ϑ_v^{AL}, ϑ_i^{AL} for wave components transmitted on toward the load at A for a wave initially traveling in the direction of the load
- ϑ_v^{AS}, ϑ_i^{AS} for wave components transmitted on toward the source at A for a wave initially traveling in the direction of the source
- ϑ_v^{BL}, ϑ_i^{BL} for wave components transmitted on toward the load at B for a wave initially traveling in the direction of the load

The notation is similar to that used for the reflection coefficients in that it indicates the discontinuity and the direction of the transmitted component.

For the example transformer, the complex-valued voltage transmission coefficients are as follows:

$$\tau_v^{AL} = 1 + \rho_v^{AS} = 1.0 + (-0.5) = 0.5 / 0$$

$$\tau_v^{AS} = 1 + \rho_v^{AL} = 1.0 + (0.5) = 1.5 / 0$$

$$\tau_v^{BL} = 1 + \rho_v^{BS} = 1.0 + (-0.5) = 0.5 / 0$$

The current transmission coefficients are:

$$\tau_i^{AL} = 1 + \rho_i^{AS} = 1.0 + (0.5) = 1.5 / 0$$

$$\tau_i^{AS} = 1 + \rho_i^{AL} = 1.0 + (-0.5) = 0.5 / 0$$

$$\tau_i^{BL} = 1 + \rho_i^{BS} = 1.0 + (0.5) = 1.5 / 0$$

VOLTAGE WAVE ANALYSIS

For a voltage wave proceeding from the source and traveling toward the load, the wave components at various instances of time are computed. At the first time snap shot (t equals t_1), a point of this wave is incident on A. This first point is defined as the phase reference for all subsequent wave components (this is different than in [3] where each point that is examined becomes the new phase reference).

For the first point on the wave (designated as E_1^{NL}), let the magnitude be 1.0 volt, and the phase be 0 degrees. Because of the impedance discontinuity at A, E_1^{NL} is partially reflected and partially transmitted. The individual complex-valued wave components are:

$t=t_1$ $E_1^{NL} = 1.0/0^o$ $E_1^{RS} = \Delta_v^{AS} E_1^{NL} = 0.5/180^o$ $E_1^{TL} = \vartheta_v^{AL} E_1^{NL} = 0.5/0^o$

where E_1^{RS} is the component reflected at A and traveling back towards the source and E_1^{TL} is the component transmitted at A and traveling on toward the load. The notation for each wave component indicates the time snap-shot, the type (N = “incident”, R = “reflected”, and T = “transmitted”) and the direction (S = “source” and “L = “load”) in which it is going. A side view of the lines at A with the three components is depicted in Figure 2. Each arrow in the figure indicates a component’s direction of travel and approximate relative magnitude. Relative phase is indicated by a “+” or a “-” over the arrow.

Note that E_1^{RS} is reduced in magnitude and inverted in phase with respect to E_1^{NL}. E_1^{TL} is also reduced in magnitude but does not change phase. The magnitude and phase of each component are consequences of the transition of impedance from high to low at A.

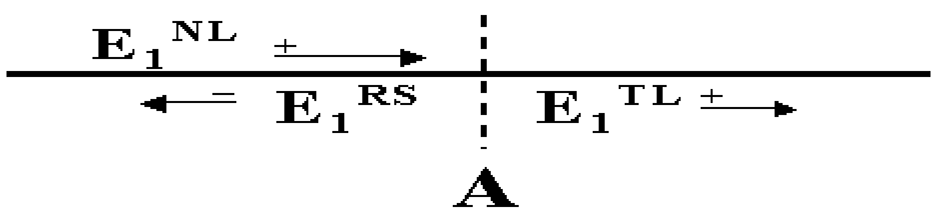

Figure 2 - Wave Components at Discontinuity A ($t = t_1$)

For now, the reflected component E_1^{RS} is ignored. As will be shown, subsequent components reflected at A are essentially canceled by other wave components (not E_1^{NL}).

At the second time snap shot (t equals t_2), the component E_1^{TL} has traveled the length (1/4-wavelength) of the 75-ohm section and is incident on discontinuity B. E_1^{TL} is re-named E_2^{NL}

since it is now an incident wave. At B, E_2^{NL} is partially reflected and partially transmitted because of the impedance discontinuity there. The individual components are:

t=t₂ $E_2^{NL} = E_1^{TL}$

$E_2^{NL} = 0.5/0^o$ $\quad E_2^{RS} = \Delta_v^{BS} E_2^{NL} = 0.25/180^o$ $\quad E_2^{TL} = \vartheta_v^{BL} E_2^{NL} = 0.25/0^o$

E_2^{RS} travels back towards A from B, and E_2^{TL} proceeds on toward the load where it is dissipated in the terminal resistance R_A.

At time equals t_3, E_2^{RS} has traveled back along the 75-ohm section and is incident on discontinuity A. However, by the time this component (re-designated to E_3^{NS}) arrives back at A, a second point on the wave from the source is arriving at A. Since the 75-ohm section is 1/4-wavelength long, the second point lags the first point by exactly 180 degrees (assuming instantaneous reflections and transmissions). Thus, there are two voltage waves incident on A, one (E_3^{NS}) traveling in the direction of the source, and one (designated E_3^{NL}) traveling in the direction of the load. See Figure 3.

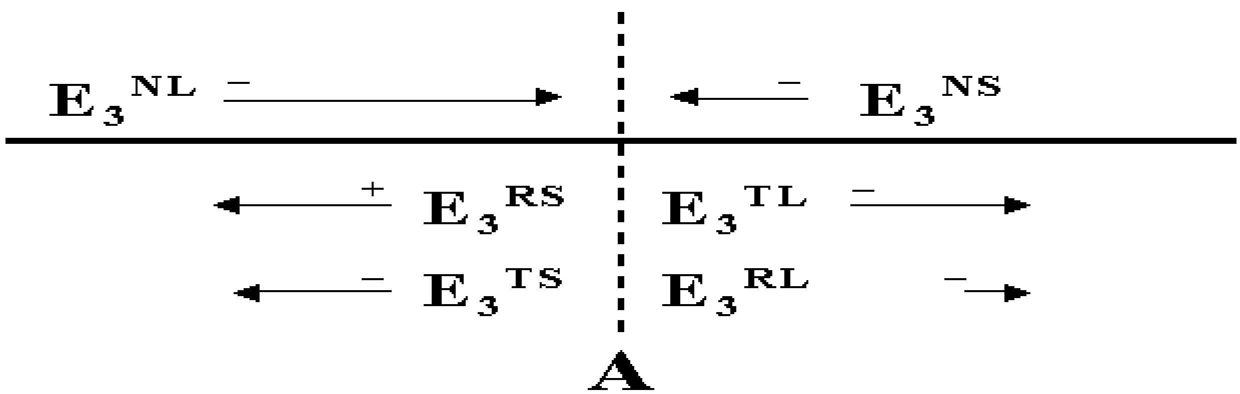

Figure 3 - Wave Components at Discontinuity A (t = t_3)

Each of these waves is partially reflected and partially transmitted at A, resulting in the following:

t=t₃ $E_3^{NS} = E_2^{RS}$

$E_3^{NS} = 0.25/180^o$ $\quad E_3^{RL} = \Delta_v^{AL} E_3^{NS} = 0.125/180^o$ $\quad E_3^{TS} = \vartheta_v^{AS} E_3^{NS} = 0.375/180^o$

$E_3^{NL} = 1.0/180^o$ $\quad E_3^{RS} = \Delta_v^{AS} E_3^{NL} = 0.5/0^o$ $\quad E_3^{TL} = \vartheta_v^{AL} E_3^{NL} = 0.5/180^o$

Relative to E_3^{NS}, E_3^{RL} is reduced in magnitude whereas E_3^{TS} is increased. Both components maintain the same phase. The magnitude and phase of each component are due to the low-to-high impedance transition at A.

Since E_3^{TS} and E_3^{RS} are traveling in the same direction (toward the source) and are exactly out of phase with each other, they destructively interfere with each other at every point along the 225-ohm section. The net backward wave at A then is:

$$E_3^{TS} + E_3^{RS} = 0.375/180^o + 0.5/0^o = 0.125/0^o$$

Similarly, E_3^{RL} and E_3^{TL} are traveling in the same direction (toward the load). However, since they are in phase with each other, they reinforce each other at every point along the 75-ohm section. The net forward wave at A is:

$$E_3^{RL} + E_3^{TL} = 0.125/180^o + 0.5/180^o = 0.625/180^o$$

Note that the waves proceeding toward the source almost cancel, but the waves proceeding toward the load add together. This process defines the essential operation of the quarter-wave transformer. However, the values of the net backward and forward waves will be examined at two additional time snap shots.

At time equals t_4, the net sum of E_3^{RL} and E_3^{TL} is now the voltage wave that is incident on B. Re-designating this as E_4^{NL}, the components are:

t=t₄ $E_4^{NL} = E_3^{RL} + E_3^{TL}$

$E_4^{NL} = 0.625/180^o$ $\quad$ $E_4^{RS} = \Delta_v^{BS} E_4^{NL} = 0.313/0^o$ $\quad$ $E_4^{TL} = \vartheta_v^{BL} E_4^{NL} = 0.313/180^o$

At time equals t_5, E_5^{NS} is incident on A. At this time instance, a third point on the wave is arriving at A from the direction of the source. This third point lags the second point by 180 degrees, and hence lags the first point by 360 degrees (i.e., a full cycle). Therefore, the third point is at 0 degrees with respect to the first point. Each of these waves is partially reflected and partially transmitted:

t=t₅ $E_5^{NS} = E_4^{RS}$

$E_5^{NS} = 0.313/0^o$ $\quad$ $E_5^{RL} = \Delta_v^{AL} E_5^{NS} = 0.156/0^o$ $\quad$ $E_5^{TS} = \vartheta_v^{AS} E_5^{NS} = 0.469/0^o$

$E_5^{NL} = 1.0/0^o$ $\quad$ $E_5^{RS} = \Delta_v^{AS} E_5^{NL} = 0.5/180^o$ $\quad$ $E_5^{TL} = \vartheta_v^{AL} E_5^{NL} = 0.5/0^o$

The net toward the source is: $\quad$ $E_5^{TS} + E_5^{RS} = 0.469/0^o + 0.5/180^o = 0.0313/180^o$

and the net toward the load is : $\quad$ $E_5^{RL} + E_5^{TL} = 0.156/0^o + 0.5/0^o = 0.656/0^o$

Therefore, after only one full cycle, the magnitude of the net backward wave proceeding toward the source from A has been reduced to 0.0313. This is due only to the partial cancellation of the two wave components E_5^{TS} and E_5^{RS}.

At subsequent time snap shots, the net backward voltage wave at A continues to decrease, whereas the net forward voltage wave continues to build. Eventually, steady-state conditions are reached, at which time the net waves on each section of line are essentially constant.

The voltage wave components appear to converge to the following values:

- backward toward the source from A: $0.0/180^o$
- forward toward B from A: $0.666\ldots/0^o$
- forward toward the termination from B: $0.333\ldots/180^o$

where the notation "…" is used to indicate a repeating decimal.

At steady-state, the net reflection coefficient $\Delta!_v$ in terms of the net incident and net reflected voltage waves is given by

$$\rho'_v = \frac{E'_{refl}}{E'_{inc}}$$

where $E!_{inc}$ is the net incident voltage wave and $E!_{refl}$ is the net reflected voltage wave. The VSWR is then given by

$$VSWR = \frac{1 + |\rho'_v|}{1 - |\rho'_v|}$$

where $|\Delta!_v|$ indicates the magnitude of the reflection coefficient. Since the net reflected wave on the 225-ohm section is essentially zero at steady state, the VSWR on this section is approximately 1.0. Furthermore, since the 25-ohm section is terminated in a 25 ohm resistance, there are no reflections, and the VSWR is also 1.0. Finally, on the 75-ohm section, $\Delta!_v$ (equals Δ_v^{BS}) is -0.5, and therefore the VSWR on this section is 3.0.

CURRENT WAVE ANALYSIS

The analysis for the current wave components is similar to that for the voltage wave components. The magnitude of the current wave I_{inc} in terms of the voltage wave E_{inc} and the characteristic impedance Z_0 is given by:

$$I_{inc} = \frac{E_{inc}}{Z_0}$$

In addition, the current wave is in phase with the voltage wave. So, for an initial E_{inc} of value $1.0/0^o$ on the 225-ohm section of line, the initial I_{inc} will be $0.00444/0^o$.

Let a current wave of value $0.00444/0^o$ proceed from the source and travel toward A. At time t equals t_1, there is a partial reflection and a partial transmission due to the impedance discontinuity at A.

$t=t_1$ $I_1^{NL} = 0.00444/0^o$ $I_1^{RS} = \Delta_i^{AS} I_1^{NL} = 0.00222/0^o$ $I_1^{TL} = \vartheta_i^{AL} I_1^{NL} = 0.00666/0^o$

At the change in impedance from high to low at A, I_1^{RS} is reduced in magnitude but I_1^{TL} is increased relative to I_1^{NL}. Both I_1^{RS} and I_1^{TL} do not change phase.

I_1^{TL} travels the length of the 75-ohm section, and at time equals t_2 is incident on discontinuity B, where it is re-named I_2^{NL}. Because of the impedance discontinuity at B, I_2^{NL} is partially reflected and partially transmitted. The components are:

t=t₂ $I_2^{NL} = I_1^{TL}$

$I_2^{NL} = 0.00666/0^o$ $\quad$ $I_2^{RS} = \Delta_i^{BS} I_2^{NL} = 0.00333/0^o$ $\quad$ $I_2^{TL} = \vartheta_i^{BL} I_2^{NL} = 0.01/0^o$

The wave component I_2^{RS} travels back towards the source and the wave component I_2^{TL} proceeds on toward the load.

At time equals t_3 there are two current waves incident on A, one (I_3^{NS}) traveling in the direction of the source, and one (I_3^{NL}) traveling in the direction of the load.

t=t₃ $I_3^{NS} = I_2^{RS}$

$I_3^{NS} = 0.00333/0^o$ $\quad$ $I_3^{RL} = \Delta_i^{AL} I_3^{NS} = 0.00167/180^o$ $\quad$ $I_3^{TS} = \vartheta_i^{AS} I_3^{NS} = 0.00167/0^o$

$I_3^{NL} = 0.00444/180^o$ $\quad$ $I_3^{RS} = \Delta_i^{AS} I_3^{NL} = 0.00222/180^o$ $\quad$ $I_3^{TL} = \vartheta_i^{AL} I_3^{NL} = 0.00666/180^o$

Because of the low-to-high impedance transition at A, both I_3^{RL} and I_3^{TS} are reduced in magnitude relative to I_3^{NS}, but only I_3^{RL} is inverted in phase.

The net toward the source is: $\quad$ $I_3^{TS} + I_3^{RS} = 0.00167/0^o + 0.00222/180^o = 0.000556/180^o$

and the net toward the load is: $\quad$ $I_3^{RL} + I_3^{TL} = 0.00167/180^o + 0.00666/180^o = 0.00833/180^o$

At time equals t_4 the net sum of I_3^{RL} and I_3^{TL} (re-designated as I_4^{NS}) is now the current wave that is incident on B.

t=t₄ $I_4^{NL} = I_3^{RL} + I_3^{TL}$

$I_4^{NL} = 0.00833/180^o$ $\quad$ $I_4^{RS} = \Delta_i^{BS} I_4^{NL} = 0.00417/180^o$ $\quad$ $I_4^{TL} = \vartheta_i^{BL} I_4^{NL} = 0.0125/180^o$

At time equals t_5 there are two current waves incident on A: I_5^{NS} traveling in the direction of the source, and I_5^{NL} traveling in the direction of the load.

t=t₅ $I_5^{NS} = I_4^{RS}$

$I_5^{NS} = 0.00417/180^o$ $\quad$ $I_5^{RL} = \Delta_i^{AL} I_5^{NS} = 0.00208/0^o$ $\quad$ $I_5^{TS} = \vartheta_i^{AS} I_5^{NS} =$ $0.00208/180^o$

$I_5^{NL} = 0.00444/0^o$ $\quad$ $I_5^{RS} = \Delta_i^{AS} I_5^{NL} = 0.00222/0^o$ $\quad$ $I_5^{TL} = \vartheta_i^{AL} I_5^{NL} = 0.00666/0^o$

The net toward the source is: $\quad$ $I_5^{TS} + I_5^{RS} = 0.00208/180^o + 0.00222/0^o = 0.000139/0^o$

and the net toward the load is: $\quad$ $I_5^{RL} + I_5^{TL} = 0.00208/0^o + 0.00666/0^o = 0.00875/0^o$

As for the voltage wave components, the net backward current wave at A continues to decrease, and the net forward current wave continues to build. At steady-state, the magnitudes of the net current waves appear to converge to the following values:

- proceeding toward the source from A: $0.0/0^o$
- proceeding toward B from A: $0.00888\ldots/0^o$
- proceeding toward the termination from B: $0.0133\ldots/180^o$

IMPEDANCES

On the 75-ohm section of transmission line, notice that the forward voltage and current waves are in phase at each time snap shot. Computing the ratio of the steady-state magnitudes of the voltage and current waves results in an impedance of Z = (0.6666…)/(0.008888…) = 75.0 ohms, which is exactly the same as the characteristic impedance of the line.

Similarly, for the 25-ohm section of transmission line, the forward voltage and current waves are also in phase at each time snap shot. The voltage-current ratio on this section is Z = (0.3333…)/(0.01333…) = 25.0 ohms. Again, this value is the same as the characteristic impedance of the line.

On the 225-ohm section of transmission line, the ratio of the two zero-valued steady-state voltage and current waves is undefined.

In effect, the quarter-wave transformer has modified the magnitudes of the forward voltage and current waves so that the proper ratio (i.e., impedance) is achieved on each section of transmission line.

POWER

The power is easily computed at a point where the impedance is purely resistive. Conveniently, this occurs at the impedance discontinuities.

The power generated for this example is computed using the initial E_{inc} and the initial I_{inc}. Hence, the power delivered to the transformer is 1.0(0.00444…) = 0.00444… watt.

On the 75-ohm section of transmission line, the net forward voltage and current waves both have a zero-degree phase. Hence, the net power flow on this section is P = (0.6666…)(0.008888…) = 0.005925 watts. Notice that the power on this section of line is greater than the power delivered to the transformer.

On the 25-ohm section of transmission line, the net forward voltage and current waves both have a 180-degree phase, and so the net power flow on this section is P = (0.3333…)(0.013333…) = 0.004444… watts. Notice that the power delivered to the load is exactly the same as the power delivered to the input of the transformer.

The quarter-wave transformer delivers all the generated power to the load. The power on the transformer itself is greater than the power delivered to the load because of the reinforcement of partially transmitted and partially re-reflected waves at the junction A.

SUMMARY

A model for a lossless quarter-wave transformer from [3] is described. This model is based on standard transmission line theory and involves a superposition of partially reflected and partially transmitted waves at the transformer's impedance discontinuities. Under steady-state conditions, all voltage and current waves proceeding backwards toward the source from the input of the transformer are essentially eliminated due to wave cancellation. All the power delivered by the source to the input of the lossless transformer is delivered to the load. The ratio of the forward voltage and current waves on the transformer is identical to the characteristic impedance of the line. The same ratio on the section of transmission line connected to the load is identical to the line's characteristic impedance.

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[3] J. Kraus, and K. Carver, *Electromagnetics,* Second Edition, New York NY: McGraw-Hill, Inc., 1973, pp. 509 to 517.

[4] W. Johnson, *Transmission Lines and Networks*, New York NY: McGraw-Hill, Inc., 1950, p. 96.

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