Solution to Problem 10317
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Closely Related Triangles

10317. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ΔABC be inscribed in a circle $\mathcal{C}$ and let A', B', C' be the midpoints of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$, respectively.

(a) Prove that the incenter of ΔABC is the orthocenter of $\Delta A'B'C'$.

(b) Prove that the pedal triangle of $\Delta A'B'C'$ is homothetic to ΔABC.

SOLUTION:

(a) Let I be the incenter of ΔABC. Let A'' be the intersection of AA' with $B'C'$. B'' and C'' are defined similarly. Let $\angle BAC = \alpha$, $\angle ABC = \beta$, and $\angle ACB = \gamma$. Then, since an inscribed angle equals 1/2 of the intercepted arc, we have the following:

$$\overset{\frown}{A'B}=\overset{\frown}{A'C}=\alpha,\ \overset{\frown}{B'A}=\overset{\frown}{B'C}=\beta,\ \overset{\frown}{C'B}=\overset{\frown}{C'A}=\gamma.$$

It is now easy to see that $\angle C'A'A = \gamma/2$ and that $\angle B'C'A' = (\overset{\frown}{B'C} + \overset{\frown}{CA'})/2 = (\beta+\alpha)/2$, thus we have that $\angle C'A'A + \angle B'C'A' = (\alpha+\beta+\gamma)/2 = 180/2 = 90$. Therefore, $B'C'$ is perpendicular to AA'. The other perpendiculars are similarly proved. Thus, I is the orthocenter of $\Delta A'B'C'$.

(b) In the book *Geometry Revisited* by Coxeter and Greitzer (New Mathematics Library), we find on p. 17 the following theorem:

The orthocenter of an acute angled triangle is the incenter of its orthic triangle.

The definition of orthic triangle given on p. 9 of the above reference shows it to be equivalent to the pedal triangle. We must show that $\Delta A'B'C'$ is acute in order to use this theorem.

Without loss of generality let $\gamma \leq \beta \leq \alpha$. Note that $\alpha + \beta < 180$. Therefore $\angle B'C'A' = (\alpha+\beta)/2 < 180/2 = 90$. Since the largest angle of $\Delta A'B'C'$ is less than 90, then the triangle is acute. Since I is now the incenter of $\Delta A''B''C''$, we have that $\angle BAI = \angle B''A''I = \alpha/2$. Therefore AB is parallel to $A''B''$. The other sides are similarly parallel which makes ΔABC homothetic to $\Delta A''B''C''$.