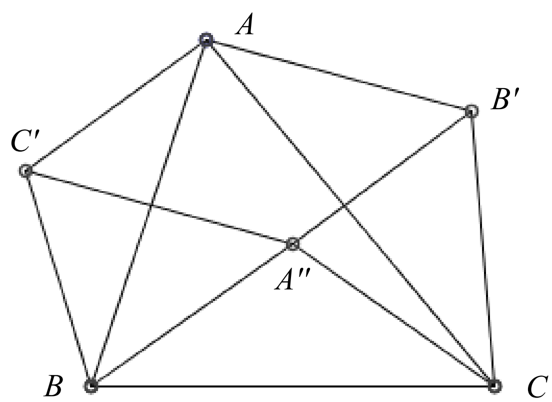
Solution to Problem E 2802
The American Mathematical Monthly

**A Parallelogram Generated from a Triangle**

E 2802 [1979, 784]. *Proposed by M. Slater, University of Bristol, England.*

Given a triangle $ABC$ (in the Euclidean plane), construct similar isosceles triangles $ABC'$ and $ACB'$ outwards on the respective bases $AB$ and $AC$, and $BCA''$ inwards on the base $BC$ (or $ABC''$ and $ACB''$ inwards and $BCA'$ outwards). Show that $AB'A''C'$ (respectively, $AB''A'C''$) is a parallelogram.




SOLUTION 1

We use the notation of David Merriell's paper, *An Application of Quasigroups to Geometry*, this Monthly 77 (1970), 44-46.

The free vertices are $C' = A\Delta B, A'' = C\Delta B$, and $B' = C\Delta A$. We now have

$$
\begin{aligned}
\frac{1}{2}(C' + B') &= \frac{1}{2}[(A\Delta B) + (C\Delta A)] \\
&= \frac{1}{2}[(A + C)\Delta(B + A)] \\
&= \frac{1}{2}[(A + C)\Delta(A + B)] \\
&= \frac{1}{1}[(A\Delta A) + (C\Delta B)] \\
&= \frac{1}{2}[A + A'']
\end{aligned}
$$

Therefore the diagonals bisect each other and $AB'A''C'$ is a parallelogram.

SOLUTION 2

Let $\beta$ equal the measure of the common base angles of the similar isosceles triangles. For example $\beta = \angle C'BA$. Let $k_1$ equal the ratio of similitude between the similar triangles $\Delta C'BA$ and $\Delta BA''C$. Let $k_2$ be the ratio of similitude between the similar triangles $\Delta B'CA$ and $\Delta A''CB$.

Now take $\Delta C'BA''$ and perform the spiral similarity $B(-\beta, k_1)$. Take $\Delta B'CA''$ and perform the spiral similarity $C(\beta, k_2)$. It is now seen that $\Delta C'BA'' \sim \Delta B'A''C$. However since $BA'' = CA''$ then these triangles are congruent. Therefore $C'A'' = B'C = B'A$ and $A''B' = BC' = C'A$ and we have opposite sides equal and $AB'A''C'$ is a parallelogram.