

Solution to Problem E 3231
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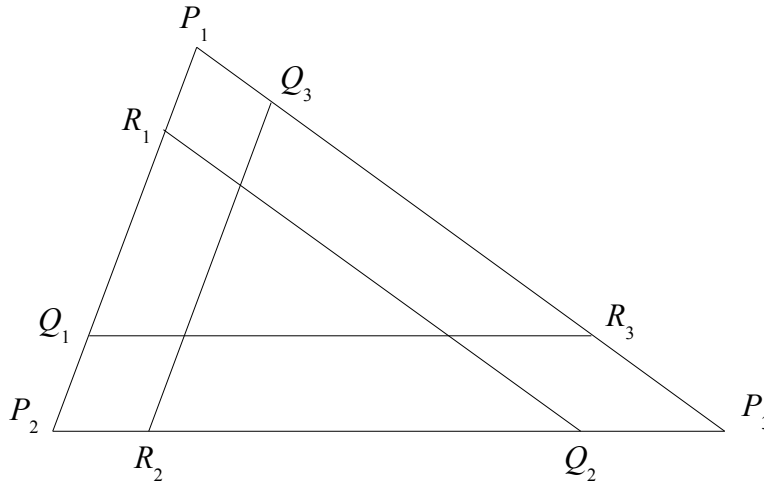
A Concurrence in the Eternal Triangle

E 3231. *Proposed by Herbert Guelicher, Muenster, West Germany.*

In a triangle $P_1P_2P_3$, let p_i be the side opposite vertex P_i , and let s_i be a line parallel to p_i (but different from p_i). Suppose that s_i divides P_iP_{i+1} in the (signed) ratio λ_i , so that if s_i meets p_{i-1} in Q_i , then $\lambda_i = P_iQ_i/Q_iP_{i+1}$. (Subscripts are taken modulo 3.) Prove that the lines s_1, s_2, s_3 are concurrent if and only if

$$\lambda_1\lambda_2\lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3) = 2.$$

SOLUTION:



Let R_i be the intersection of s_{i+1} and P_iP_{i+1} . From the parallelograms in the corners of the triangle, we see that s_1, s_2, s_3 are concurrent if

$$s_i = P_{i+1}R_{i+1} + Q_{i+1}P_{i+2} \tag{1}$$

By similar triangles we get the following relationships:

$$P_i Q_i / Q_i P_{i+1} = \lambda_i \Rightarrow P_i Q_i / P_i P_{i+1} = \lambda_i / (1 + \lambda_i) = s_i / P_{i+1} P_{i+2} \text{ so}$$

$$s_i = p_i \lambda_i / (1 + \lambda_i) \tag{2}$$

so $P_{i+1} R_{i+1} / R_{i+1} P_{i+2} = P_i Q_{i+2} / Q_{i+2} P_{i+2} = 1 / \lambda_{i+2} \Rightarrow P_{i+1} R_{i+1} / P_{i+1} P_{i+2} = 1 / (1 + \lambda_{i+2})$

$$P_{i+1} R_{i+1} = p_i / (1 + \lambda_{i+2}) \tag{3}$$

$$Q_{i+1} P_{i+2} / P_{i+1} Q_{i+1} = 1 / \lambda_{i+1} \Rightarrow Q_{i+1} P_{i+2} / P_{i+1} P_{i+2} = 1 / (1 + \lambda_{i+1}) \text{ so}$$

$$Q_{i+1} P_{i+2} = p_i / (1 + \lambda_{i+1}) \tag{4}$$

Substitute (2), (3), and (4) into (1) and reduce. This yields

$$\lambda_i / (1 + \lambda_i) = 1 / (1 + \lambda_{i+2}) + 1 / (1 + \lambda_{i+1}).$$

On eliminating fractions and collecting terms we get

$$\lambda_1 \lambda_2 \lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3) = 2.$$