Solution to Problem E 3302
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Averaging to Integers

E 3302. *Proposed by Jim Delany, California Polytechnic State University, San Luis Obispo.*

The mean and standard deviation of any 7 consecutive integers are both integers. What natural numbers greater than 1 share this property with 7?

SOLUTION: Without loss of generality, let the n consecutive integers be $1, 2, 3, \ldots, n$. This only translates the mean an integral amount and leaves the standard deviation unchanged. Let the population parameters be

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \text{and} \qquad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - n\mu^2\right)}$$

where $x_i = i$. Using the standard formulas for the sum of integers we get

$$\mu = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2}$$

from which we observe that n must be odd. Now use the sum of squares formula to gain an expression for the variance

$$\begin{aligned} \sigma^2 &= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6} - n\cdot\left(\frac{n+1}{2}\right)^2\right] \\ &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \end{aligned}$$

Getting a common denominator and simplifying we arrive at the key equation $\sigma^2 = (n^2-1)/12$ or

$$n^2 - 12\sigma^2 = 1.$$

This is the Pell equation. Since the variance cannot be 0 or 1, then the smallest solution to this equation is $n = 7$ and $\sigma = 2$. All solutions to this equation can now be generated by the well-known formulas

$$n + \sigma\sqrt{12} = (7 + 2\sqrt{12})^k \qquad k = 1, 2, 3, \ldots$$

Thus there are an infinite number of numbers that share the desired property with 7. The sequence is $7, 97, 1351, \ldots$.