Solution to Problem E 3392
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The Area of a Pedal of a Pedal Triangle

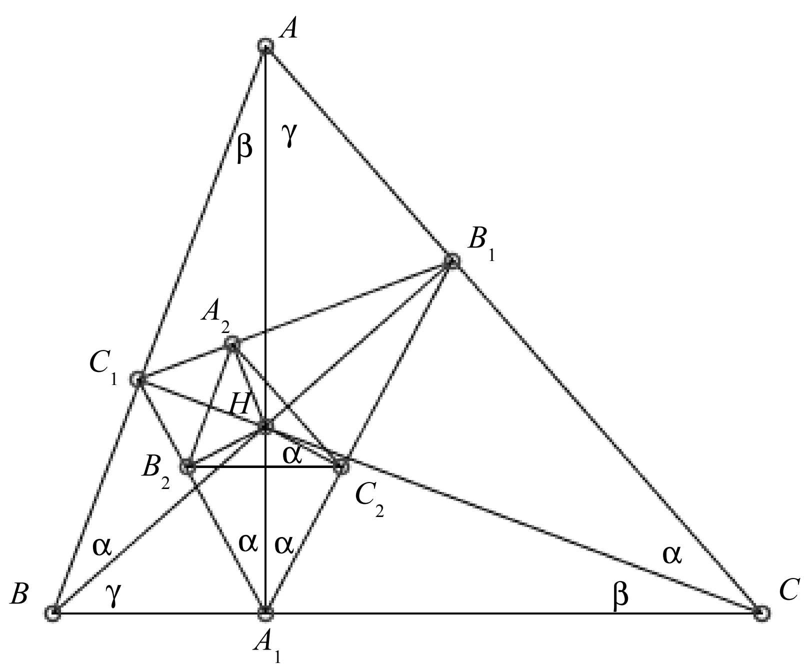
E 3392 *Proposed by Antal Bege, Miercurea-Ciuc, Romania.*

Given an acute-angled triangle ABC with orthocenter H, let A_1, B_1, C_1 be the feet of the altitudes from A, B, C, respectively, and let A_2, B_2, C_2 be the feet of the perpendiculars from H onto B_1C_1, C_1A_1, A_1B_1, respectively. Prove that

$$\text{area}(\Delta ABC) \geq 16\text{area}(\Delta A_2B_2C_2)$$

and determine when equality holds.

SOLUTION:



The orthocenter of an acute angled triangle is the incenter of its orthic triangle. Thus $HA_2 = HB_2 = HC_2 = r$, $\alpha + \beta + \gamma = 90°$, $\Delta ABC \sim \Delta A_2B_2C_2$.

Repeated use of the law of sines yields

$$\begin{array}{rcl}
\Delta HB_2C_2 \Rightarrow \dfrac{B_2C_2}{\sin(180-2\alpha)} = \dfrac{r}{\sin\alpha} & \Rightarrow & B_2C_2 = 2r\cos\alpha \\
\Delta ABC \Rightarrow \dfrac{BC}{\sin(\beta+\gamma)} = \dfrac{AB}{\sin(\alpha+\beta)} & \Rightarrow & BC = \dfrac{AB\cos\alpha}{\cos\gamma} \\
\Delta ABH \Rightarrow \dfrac{AB}{\sin(180-\alpha-\beta)} = \dfrac{BH}{\sin\beta} & \Rightarrow & AB = \dfrac{BH\cos\gamma}{\sin\beta}
\end{array}$$

By right triangle trigonometry,

$$BH = \frac{A_1H}{\sin\gamma} \text{ and } A_1H = \frac{r}{\sin\alpha}$$

Repeated substitution gives us

$$\frac{BC}{B_2C_2} = \frac{1}{2\sin\alpha\sin\beta\sin\gamma}.$$

Therefore

$$\text{area}(\Delta ABC) = \left(\frac{1}{2\sin\alpha\sin\beta\sin\gamma}\right)^2 \cdot \text{area}(\Delta A_2B_2C_2).$$

Since α, β, γ are acute angles, we can use the inequality on arithmetic and geometric means to get the following:

$$\sin\alpha\sin\beta\sin\gamma \leq \left(\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\right)^3$$

with equality if and only if $\sin\alpha = \sin\beta = \sin\gamma$. However this implies that $\alpha = \beta = \gamma = 30°$.

$$\begin{aligned} \sin\alpha\sin\beta\sin\gamma &\leq \left(\frac{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}{3}\right)^3 = \frac{1}{8} \\ \frac{1}{\sin\alpha\sin\beta\sin\gamma} &\geq 8 \end{aligned}$$

Dividing both sides by 2 and squaring yields the desired result.