Solution to Problem 376
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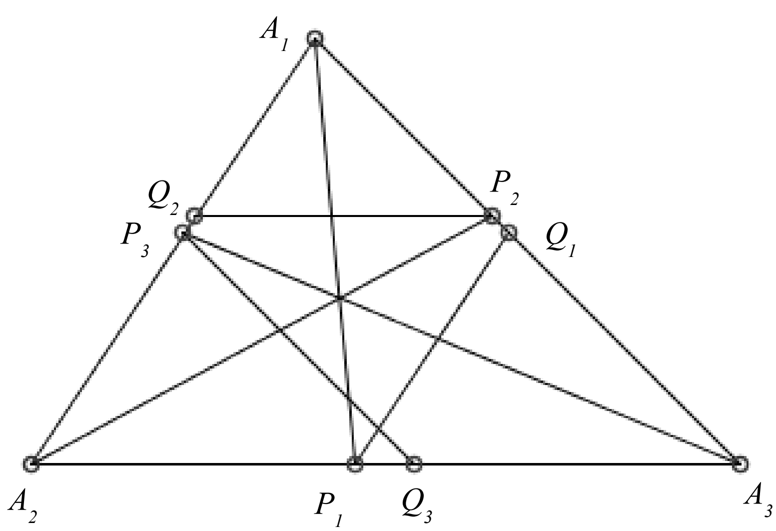
Reciprocals of Triangle Sides

376. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

In triangle $A_1A_2A_3$, Let $P_i \in \overline{A_{i+1}A_{i+2}}$ and $Q_i \in \overline{A_{i-1}A_i}$, $i = 1, 2, 3$ ($A_4 = A_1, A_5 = A_2, A + 0 = A_3$), such that $\overline{A_iP_i}$ bisects $\angle A_i$ and $\overline{P_iQ_i} \parallel \overline{A_iA_{i+1}}$. Prove that

$$\sum_{i=1}^{3} \frac{1}{P_iQ_i} = 2\sum_{i=1}^{3} \frac{1}{A_iA_{i+1}}$$

SOLUTION



By similar triangles, $\dfrac{P_iQ_i}{P_iA_{i+2}} = \dfrac{A_iA_{i+1}}{A_{i+1}A_{i+2}}$. Thus, $\dfrac{1}{P_iQ_i} = \dfrac{A_{i+1}A_{i+2}}{(P_iA_{i+2})(A_iA_{i+1})}$

Also, since an angle bisector divides the side it hits into the ratio of the two adjacent sides, we have

$$\frac{A_iA_{i+1}}{A_iA_{i+2}} = \frac{A_{i+1}P_i}{P_iA_{i+2}} \qquad \text{and} \qquad \frac{A_{i+1}A_{i+2}}{P_iA_{i+2}} = \frac{A_iA_{i+1} + A_iA_{i+2}}{A_iA_{i+2}}.$$

Using the above, we get

$$\begin{aligned}
\sum_{i=1}^{3} \frac{1}{P_iQ_i} &= \sum_{i=1}^{3} \frac{A_{i+1}A_{i+2}}{(P_iA_{i+2})(A_iA_{i+1})} \\
&= \sum_{i=1}^{3} \frac{A_iA_{i+1} + A_iA_{i+2}}{(A_iA_{i+2})(A_iA_{i+1})} \\
&= \sum_{i=1}^{3} \left(\frac{1}{A_iA_{i+2}} + \frac{1}{A_iA_{i+1}} \right) \\
&= 2\sum_{i=1}^{3} \frac{1}{A_iA_{i+1}}
\end{aligned}$$