Solution to Problem 402
The College Mathematics Journal

Sums of Roots

402. *Proposed by Norman Shaumberger, Bronx Community College, Bronx, NY.*

Show that

$$\sqrt[5]{\frac{11+5\sqrt{5}}{2}}+\sqrt[5]{\frac{11-5\sqrt{5}}{2}}=\sqrt[7]{\frac{29+13\sqrt{5}}{2}}+\sqrt[7]{\frac{29-13\sqrt{5}}{2}}.$$

SOLUTION:

Let F_m and L_m denote the Fibonacci and Lucas sequences, respectively, i.e.,

$$F_0=0 \qquad F_1=1 \qquad F_m=F_{m-1}+F_{m-2} \text{ for } m\geq 2\text{; and}$$

$$L_0=2 \qquad L_1=1 \qquad L_m=L_{m-1}+L_{m-2} \text{ for } m\geq 2.$$

We prove the following more general result: If n and k are positive integers, then

$$\sqrt[n]{\frac{L_n+F_n\sqrt{5}}{2}}+(-1)^{k+1}\sqrt[k]{\frac{L_k-F_k\sqrt{5}}{2}}=1$$

Proof. It is well known that if $a=(1+\sqrt{5})/2$ and $b=(1-\sqrt{5})/2$, then $L_m=a^m+b^m$ and $F_m=(a^m-b^m)/\sqrt{5}$, from which the desired result follows by noting that $\sqrt[n]{a^n}=a$ and $\sqrt[k]{b^k}=(-1)^{k+1}b$.