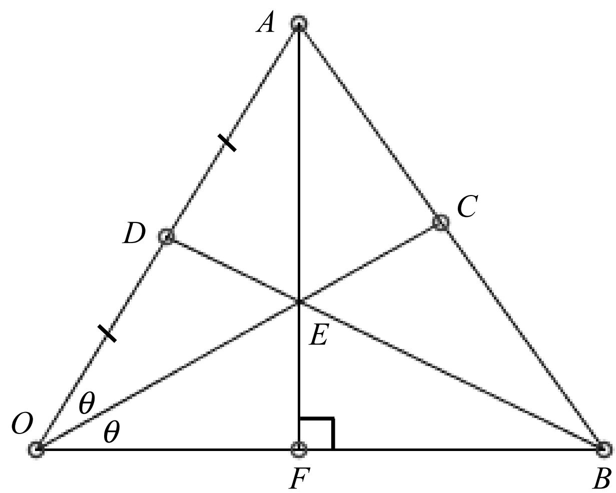
Solution to Problem 455
The College Mathematics Journal

A Triangle with Concurrent Median, Altitude, and Angle Bisector

455. *Proposed by R. S. Tiberio, Natick, MA.*

Characterize those triangles such that the angle bisector from one vertes, the median from a second vertex and the altitude from the third vertex are concurrent.



SOLUTION

Choose O as an origin. We will use a vector approach. Let $|\vec{a}| = |\overrightarrow{OA}| = a, |\vec{b}| = |\overrightarrow{OB}| = b$. The position vector for the angle bisector from O is given by

$$\vec{c} = \frac{b\vec{a} + a\vec{b}}{a+b}$$

The median is $\overrightarrow{BD} = \vec{d} - \vec{b} = \dfrac{1}{2}\vec{a} - \vec{b}$.

Now we generate the position vector to the point E. Since $\vec{d} = (1/2)\vec{a}$, we get

$$\begin{aligned} \vec{c} &= \frac{2b\vec{d} + a\vec{b}}{a+b} \\ \left(\frac{a+b}{a+2b}\right)\vec{c} &= \frac{2b\vec{d} + a\vec{b}}{a+2b} = \vec{e} \end{aligned}$$

We want $\overrightarrow{AE} \perp \vec{b}$ or $(\vec{e} - \vec{a}) \cdot \vec{b} = 0$. Thus

$$\begin{aligned} \left(\frac{b\vec{a} + a\vec{b}}{a+2b} - \vec{a}\right) \cdot \vec{b} &= 0 \\ \left(a\vec{b} - (a+b)\vec{a}\right) \cdot \vec{b} &= 0 \\ (a+b)\vec{a} \cdot \vec{b} &= ab^2 \\ ab\cos 2\theta &= \frac{ab^2}{a+b} \\ \cos 2\theta &= \frac{b}{a+b} \end{aligned}$$

Therefore, if the angle bisector from O, altitude from A and median from B are concurrent then the triangle is described by sides a and b and included angle of

$$2\theta = \cos^{-1}\left(\frac{b}{a+b}\right)$$