Solution to Problem 460
The College Mathematics Journal

$$\mathbf{y_{n+1} = a y_n^x}$$

460. *Proposed by William V. Webb, Akron, OH.*

Solve the difference equation: $y_{n+1} = ay_n^x$ $(n = 0, 1, \ldots)$, where $a > 0$; $x \neq 0$; and $y_0 > 0$.

SOLUTION:

An examination of cases leads to the following observations:

$$\begin{aligned}
y_1 &= ay_0^x \\
y_2 &= ay_1^x = a(ay_0^x)^x = a^{x+1}y_0^{x^2} \\
y_3 &= ay_2^x = a\left(a^{x+1}y_0^{x^2}\right)^x = a^{x^2+x+1}y_0^{x^3} \\
y_4 &= ay_3^x = a\left(a^{x^2+x+1}y_0^{x^3}\right)^x = a^{x^3+x^2+x+1}y_0^{x^4}
\end{aligned}$$

Since $x \neq 0$ or 1, we may express the cyclotomic polynomials more compactly and the general solution can be written as

$$y_n = a^{\frac{x^n-1}{x-1}} \cdot y_0^{x^n}$$