Solution to Problem 467
The College Mathematics Journal

$\cos\mathbf{A} : \cos\mathbf{B} : \cos\mathbf{C} = \mathbf{2 : 9 : 12} \Rightarrow \mathbf{a : b : c = 6 : 5 : 4}$

467. *Proposed by Stanley Rabinowitz, Westford, MA.*

The cosines of the angles of a triangle are in the ratio 2:9:12. Find the ratio of the sides of the triangle.

SOLUTION:

Let the angles of the triangle be α, β, and γ with sides opposite these angles of a, b, and c. We then have $\cos\alpha = 2k$, $\cos\beta = 9k$, and $\cos\gamma = 12k$ for some constant k. If $k < 0$ then all three angles are obtuse. Since this is clearly impossible for a triangle, we have that $k > 0$. Starting with $\alpha + \beta + \gamma = 180°$, we derive the following sequence of trig identities.

$$\begin{aligned}
\alpha + \beta &= 180 - \gamma \\
\cos(\alpha + \beta) &= \cos(180 - \gamma) \\
\cos\alpha\cos\beta - \sin\alpha\sin\beta &= -\cos\gamma \\
\cos\alpha\cos\beta + \cos\gamma &= \sin\alpha\sin\beta \\
\cos^2\alpha\cos^2\beta + 2\cos\alpha\cos\beta\cos\gamma + \cos^2\gamma &= (1 - \cos^2\alpha)(1 - \cos^2\beta) \\
\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma &= 1
\end{aligned}$$

Pluging in our values for the cosines yields the following equation:

$$(2k)^2 + (9k)^2 + (12k)^2 + 2(2k)(9k)(12k) = 1$$
or
$$432k^3 + 229k^2 - 1 = 0.$$

Descartes' rule of signs shows that this equation has exactly 2 negative roots and one positive root. The positive root is $k = 1/16$ exactly. This now gives us $\cos\alpha = 1/8$, $\cos\beta = 9/16$, and $\cos\gamma = 3/4$. We can now use the law of sines to get

$$\frac{a}{\sqrt{1-\left(\frac{1}{8}\right)^2}} = \frac{b}{\sqrt{1-\left(\frac{9}{16}\right)^2}} = \frac{c}{\sqrt{1-\left(\frac{3}{4}\right)^2}}$$

This quickly simplifies to

$$\frac{a}{6} = \frac{b}{5} = \frac{c}{4}$$

So the ratio of sides is $a : b : c = 6 : 5 : 4$.