Solution to Problem 477
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Another Trigonometric Inequality

477. *Proposed by Norman Schaumberger, Hofstra University, Hempstead, NY.*

In triangle ABC, let a, b, and c be the lengths of the sides opposite angles of measure α, β, and γ, respectively; and let p denote the perimeter. Prove that

$$\frac{a\sin\alpha + b\sin\beta + c\sin\gamma}{p} \geq (\sin\alpha\sin\beta\sin\gamma)^{1/3}$$

SOLUTION: Let $R =$ the circumradius of ΔABC. Then by the extended law of sines we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \frac{a+b+c}{\sin\alpha+\sin\beta+\sin\gamma} = 2R.$$

Solving for a, b, c, and the perimeter, we can plug into the left hand side to get

$$\frac{a\sin\alpha + b\sin\beta + c\sin\gamma}{p} = \frac{2R\sin^2\alpha + 2R\sin^2\beta + 2R\sin^2\gamma}{2R(\sin\alpha+\sin\beta+\sin\gamma)}.$$

We can reduce by $2R$, multiply by $3/3$, and use both the Cauchy inequality and the arithmetic-geometric mean inequality to get

$$\frac{3\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)}{3\left(\sin\alpha+\sin\beta+\sin\gamma\right)} \geq \frac{\left(\sin\alpha+\sin\beta+\sin\gamma\right)^2}{3\left(\sin\alpha+\sin\beta+\sin\gamma\right)} \geq \frac{\sin\alpha+\sin\beta+\sin\gamma}{3} \geq (\sin\alpha\sin\beta\sin\gamma)^{1/3}.$$