Solution to Problem 486
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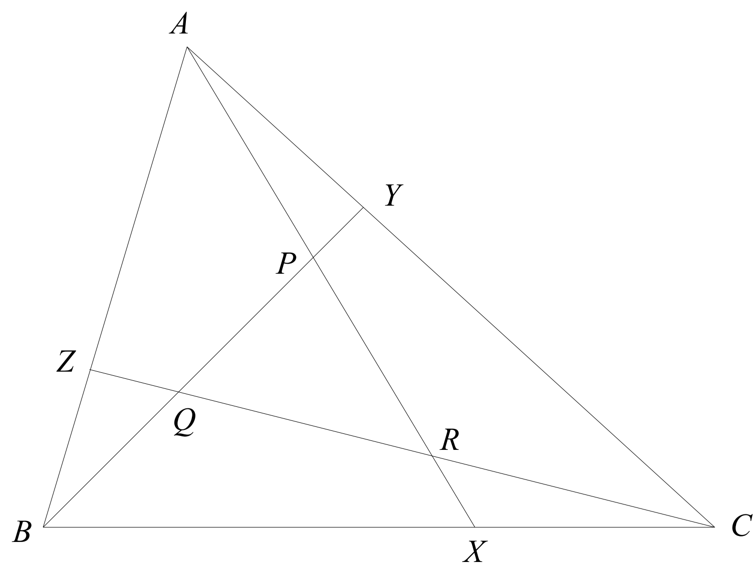
Constant Ratios in a Triangle

486. *Proposed by Jyotirmoy Sarkar, University of Michigan, Ann Arbor, MI.*

Let X, Y and Z be points on sides BC, CA and AB, respectively, of triangle ABC such that $AZ : ZB = BX : XC = CY : YA$. Suppose that AX intersects BY at P, BY intersects CZ at Q, and that CZ intersects AX at R. If P, Q, and R do not coincide, prove that

$$AR : RP : PX = BP : PQ : QY = CQ : QR : RZ.$$

SOLUTION:



Let $AZ : ZB = BX : XC = CY : YA = 1/\rho$. Let $\vec{a}$ be a vector from an arbitrary origin to point A, etc. We then have

$$\vec{z} = \frac{\rho\vec{a} + \vec{b}}{\rho + 1} \qquad \vec{x} = \frac{\rho\vec{b} + \vec{c}}{\rho + 1} \qquad \vec{y} = \frac{\rho\vec{c} + \vec{a}}{\rho + 1} \tag{1}$$

From the first two equations of (1) we eliminate $\vec{b}$ and rearrange to get

$$\frac{\rho^2\vec{a} + (\rho + 1)\vec{x}}{\rho^2 + \rho + 1} = \frac{(\rho^2 + \rho)\vec{z} + \vec{c}}{\rho^2 + \rho + 1} = \vec{r} \tag{2}$$

Similarly, we take the last two equations of (1), eliminate $\vec{c}$ and rearrange to get

$$\frac{\rho^2\vec{b} + (\rho + 1)\vec{y}}{\rho^2 + \rho + 1} = \frac{(\rho^2 + \rho)\vec{x} + \vec{a}}{\rho^2 + \rho + 1} = \vec{p} \tag{3}$$

We now take $\vec{r}$ equal to the left side of (2), $\vec{p}$ equal to the right side of (3), eliminate $\vec{x}$ and rearrange to get

$$\vec{r} = \frac{(\rho - 1)\vec{a} + \vec{p}}{\rho}$$

or equivalently, $AR : RP = 1 : (\rho - 1) = (\rho + 1) : (\rho^2 - 1)$. In like manner we take the same two equations but this time eliminate $\vec{a}$ and rearrange to get

$$\vec{p} = \frac{\vec{r} + (\rho^2 - 1)\vec{x}}{\rho^2}$$

or equivalently, $RP : PX = (\rho^2 - 1) : 1$. Combining these last two results we arrive at

$$AR : RP : PX = (\rho + 1) : (\rho^2 - 1) : 1.$$

The same expression for $BP : PQ : QY$ and $CQ : QR : RZ$ are derived similarly.