Solution to Problem 503
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Counting "Diophantine" Triangles

503. *Proposed by Louis R. Bragg and Jerrold W. Grossman, Oakland University, Rochester, MI.*

How many different triangles are there with integral sides of length at most n?

SOLUTION:

Let $T(n)$ be the required number of triangles. Let $f(n)$ be the number of triangles with integral sides having longest side of length exactly n. Then

$$T(n) = \sum_{k=1}^{n} f(k).$$

The problem of finding $f(n)$ has appeared in numerous places (for example, the 1949 Stanford University Competitive Mathematics Examination). The result can be expressed in a number of ways. Here are two.

$$f(n) = \left\lfloor \left(\frac{n+1}{2} \right)^2 \right\rfloor$$
$$f(n) = \frac{n^2 + 2n}{4} + \frac{1 - (-1)^n}{8}.$$

Using standard summation formulas and collecting terms, we arrive at the following formula for T:

$$T(n) = \left\lfloor \frac{(n+1)(n+3)(2n+1)}{24} \right\rfloor.$$