Solution to Problem 515
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A Parallelogram Characterization

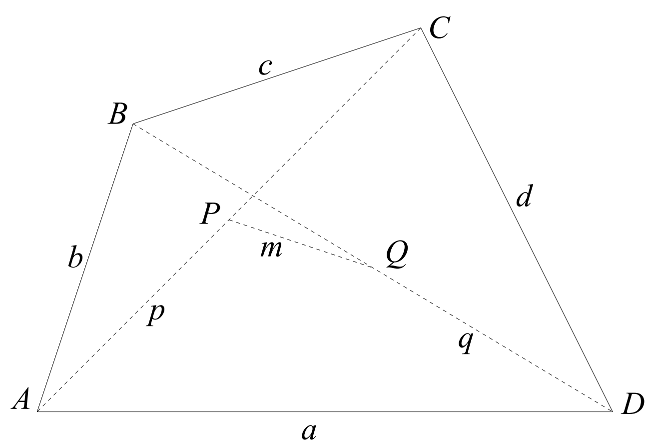
515. *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, TN.*

Let $ABCD$ be a quadrilateral with $AB = b$, $BC = c$, $CD = d$, and $DA = a$. Prove that $ABCD$ is a parallelogram if and only if

$$ab\cos A + bc\cos B + cd\cos C + da\cos D = 0.$$

SOLUTION:

Let $AC = p$, $BD = q$, and $PQ = m$ where P is the midpoint of AC and Q is the midpoint of BD.



($\Rightarrow$)

If $ABCD$ is a parallelogram, then $a = c$, $b = d$, $A = C$, $B = D$, and $A + B = 180°$. Substituting, we have

$$ab\cos A + bc\cos B + cd\cos C + da\cos D = 2ab(\cos A + \cos B) = 0$$

since A and B are supplementary.

($\Leftarrow$)

By the law of cosines we have

$$\begin{aligned} q^2 &= a^2 + b^2 - 2ab\cos A \\ q^2 &= c^2 + d^2 - 2cd\cos C \\ p^2 &= b^2 + c^2 - 2bc\cos B \\ p^2 &= d^2 + a^2 - 2da\cos D \end{aligned}$$

Adding these gives

$$2p^2 + 2q^2 = 2a^2 + 2b^2 + 2c^2 + 2d^2 - 2(ab\cos A + bc\cos B + cd\cos C + da\cos D).$$

By hypothesis, the parenthetical expression is zero. Dividing by 2 yields

$$p^2 + q^2 = a^2 + b^2 + c^2 + d^2.$$

However, in any quadrilateral we have

$$p^2 + q^2 + 4m^2 = a^2 + b^2 + c^2 + d^2.$$

(This result appears on p. 56, problem 2 in *Geometry Revisited* by Coxeter and Greitzer, New Mathematics Library.) Therefore, $4m^2 = 0$ and $m = 0$. This implies that P and Q are coincident. Since the diagonals of this quadrilateral bisect each other, it is a parallelogram.