Solution to Problem 531
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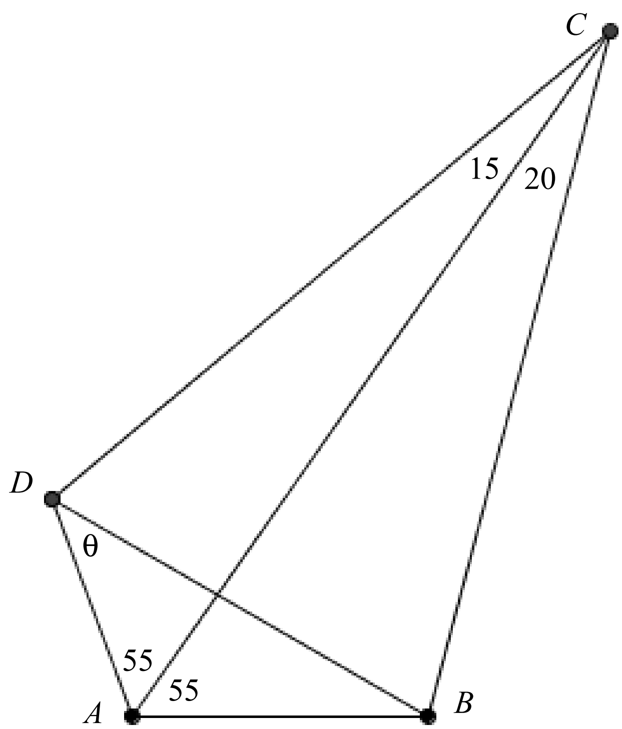
Solving a Quadrilateral

531. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta, Edmonton, Canada.*

If A, B, C, and D are consecutive vertices of a quadrilateral such that $\angle DAC = 55° = \angle CAB$, $\angle ACD = 15°$, and $\angle BCA = 20°$, determine $\angle ADB$.

SOLUTION: Let $\theta = \angle ADB$.



Using the law of sines on ΔADC, ΔABC and ΔABD yields

$$\frac{AD}{\sin 15} = \frac{AC}{\sin 110}, \frac{AC}{\sin 105} = \frac{AB}{\sin 20}, \frac{AD}{\sin(70-\theta)} = \frac{AB}{\sin\theta}$$

On eliminating sides, we are left with

$$\frac{\sin 105 \cdot \sin 15}{\sin 110 \cdot \sin 20} = \frac{\sin(70-\theta)}{\sin\theta}$$

Using the "sum and product" identities on the left side, we get

$$\frac{\frac{1}{2}(\cos 90 - \cos 120)}{\frac{1}{2}(\cos 90 - \cos 130)} = \frac{\sin(70-\theta)}{\sin\theta}$$

Simplifying and cross multiplying gives

$$\sin\theta \cdot \cos 120 = \sin(70-\theta) \cdot \cos 130$$

Use the "sum and product" identities again to give

$$\frac{1}{2}\left(\sin(\theta+120) + \sin(\theta-120)\right) = \frac{1}{2}\left(\sin(200-\theta) + \sin(-60-\theta)\right)$$
or
$$\sin(\theta+120) - \sin(120-\theta) = \sin(200-\theta) - \sin(60+\theta)$$

Since $(120-\theta)$ and $(60+\theta)$ are supplementary, their sines are equal. Thus we are left with

$$\sin(\theta+120) = \sin(200-\theta).$$

Since $\angle DAB = 110$, then θ is an acute angle. Therefore,

$$\theta + 120 = 200 - \theta$$
and
$$\theta = 40°.$$