

**A Plethora of Pairs of Primitive Pythagorean Triples**

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Consider the following pair of primitive Pythagorean triples: (3,4,5) and (5,12,13). Observe that (i) the legs of the second are, respectively, the hypotenuse and the product of the legs of the first and (ii) a leg and the hypotenuse of the second are consecutive integers. Are there other such pairs of primitive Pythagorean triples?

SOLUTION: Yes. There are an infinite number of them.

Let the first triple be  $(a, b, c)$ . Then the conditions of the problem require that the second triple be  $(c, ab, ab + 1)$ . Thus,

$$\begin{aligned}c^2 + (ab)^2 &= (ab + 1)^2 \\c^2 + a^2b^2 &= a^2b^2 + 2ab + 1 \\c^2 &= 2ab + 1\end{aligned}$$

Since  $c^2 = a^2 + b^2$ , this implies

$$\begin{aligned}a^2 + b^2 &= 2ab + 1 \\(a - b)^2 &= 1\end{aligned}$$

Thus the two legs of the original triangle must differ by 1. It is a routine textbook exercise to show that if  $(x, x + 1, z)$  is a primitive Pythagorean triple, then so is  $(3x + 2z + 1, 3x + 2z + 2, 4x + 3z + 2)$ . [See, for example, *Elementary Number Theory* by David M. Burton, p. 250 #12(a).] We can now generate an infinite number of such triples from the original pair.

$$\begin{array}{ccc} (3,4,5) & \text{and} & (5,12,13) \\ (20,21,29) & \text{and} & (29,420,421) \\ (119,120,169) & \text{and} & (169,14280,14281) \\ \dots & & \dots \end{array}$$