

Solution to Problem 558  
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**A Corollary of Ceva's Theorem**

**558.** *Proposed by Jiro Fukuta, Shinsei-cho, Gifu-ken, Japan.*

Prove that for any triangle  $ABC$ , there exists one and only one set of points  $D, E, F$  satisfying:

- (a)  $D$  lies on side  $BC$ ,  $E$  lies on side  $CA$ , and  $F$  lies on side  $AB$ .
- (b)  $\overline{EA} + \overline{AF} = \overline{BC}$ ;  $\overline{FB} + \overline{BD} = \overline{CA}$ ; and  $\overline{DC} + \overline{CE} = \overline{AB}$ ; and
- (c)  $AD$ ,  $BE$ , and  $CF$  are concurrent.

SOLUTION:

Existence

Let  $s =$  the semi-perimeter of the triangle.

Locate  $D$  so that  $\overline{AB} + \overline{BD} = s = \overline{AC} + \overline{DC}$ .

Locate  $E$  so that  $\overline{AB} + \overline{AE} = s = \overline{BC} + \overline{EC}$ .

Locate  $F$  so that  $\overline{AC} + \overline{AF} = s = \overline{BC} + \overline{FB}$ .

These equations easily imply that

$$\overline{AF} = \overline{DC}, \overline{AE} = \overline{BD}, \text{ and } \overline{FB} = \overline{EC} \tag{1}$$

Therefore,  $\overline{EA} + \overline{AF} = \overline{BD} + \overline{DC} = \overline{BC}$ . Similarly,  $\overline{FB} + \overline{BD} = \overline{EC} + \overline{AE} = \overline{AC}$  and  $\overline{DC} + \overline{CE} = \overline{AF} + \overline{FB} = \overline{AB}$ . Also from (1) we have that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 \quad (2)$$

and thus by Ceva's Theorem,  $AD$ ,  $BE$ , and  $CF$  are concurrent. This point is the well-known Nagel point (see Eves, A Survey of Geometry, p. 71 #3).

Uniqueness

Suppose a different set of points  $D'$ ,  $E'$ , and  $F'$  satisfy the conditions of the problem. Suppose also (without loss of generality) that  $\overline{AF'} = \overline{AF} - x$  for some  $x > 0$ . Then the conditions of the problem require that  $\overline{F'B} = \overline{FB} + x$ . However, since  $\overline{AF'} + \overline{AE'} = \overline{BC}$  we also have that  $\overline{AE'} = \overline{AE} + x$ . Similarly,  $\overline{BD'} = \overline{BD} - x$ ,  $\overline{D'C} = \overline{DC} + x$ , and  $\overline{CE'} = \overline{CE} - x$ . But then we get

$$\frac{\overline{AF'}}{\overline{F'B}} \cdot \frac{\overline{BD'}}{\overline{D'C}} \cdot \frac{\overline{CE'}}{\overline{E'A}} = \frac{\overline{AF} - x}{\overline{FB} + x} \cdot \frac{\overline{BD} - x}{\overline{DC} + x} \cdot \frac{\overline{CE} - x}{\overline{EA} + x} < 1$$

since each fraction has a smaller numerator and larger denominator than the corresponding fractions in (2). Thus there is no concurrence and the points  $D$ ,  $E$ , and  $F$  are unique.