

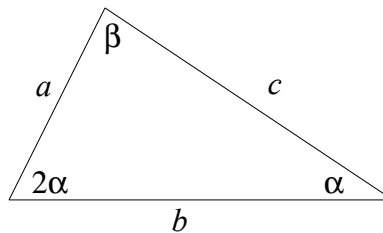
Solution to Problem 578
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Triangles with One Angle Twice Another

578. *Proposed by Robert Patenaude, College of the Canyons, Valencia, CA.*

Determine the lengths of the sides of a triangle with the properties that (i) the sides have integral length and (ii) one angle is twice as large as another.

SOLUTION:



Clearly the triangle is not equilateral. The only isosceles triangles with one angle twice another are 45-45-90 and 72-72-36. It is easy to show that in both cases all three sides cannot be integral. So we may assume that the triangle is scalene.

By the law of sines we have

$$\frac{c}{\sin 2\alpha} = \frac{a}{\sin \alpha} \text{ or } c = 2a \cdot \cos \alpha$$

From the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cdot \cos \alpha$$

$$a^2 = b^2 + c^2 - 2bc \cdot \frac{c}{2a}$$

$$a^3 = ab^2 + ac^2 - bc^2.$$

This can be regrouped and factored as

$$a(a^2 - b^2) = c^2(a - b).$$

Since the triangle is scalene we can divide by $a - b$ and get

$$a^2 + ab = c^2.$$

To solve this equation we multiply by 4 and complete the square to obtain

$$\begin{aligned}4a^2 + 4ab &= 4c^2 \\4a^2 + 4ab + b^2 &= 4c^2 + b^2 \\(2a + b)^2 &= (2c)^2 + b^2.\end{aligned}$$

This equation can now be solved by using the standard solution to the Pythagorean equation $x^2 + y^2 = z^2$ with $x = m^2 + n^2$, $y = 2mn$, and $z = m^2 - n^2$ which can be found in most number theory books. We take

$$2a + b = m^2 + n^2, 2c = 2mn, b = m^2 - n^2.$$

This yields

$$a = n^2, b = m^2 - n^2, c = mn$$

with the usual requirements that $m > n > 0$ and m, n are relatively prime. However the triangle inequality gives additional restrictions since

$$\begin{aligned}a + b &> c \\n^2 + mn &> m^2 - n^2 \\n(n + m) &> (m - n)(m + n) \\2n &> m > n\end{aligned}$$

which implies that $n > 1$. So the smallest solution is $m = 3, n = 2$ which yields $a = 4, b = 5, c = 6$.