

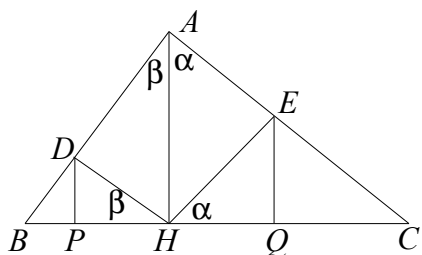
Solution to Problem 599
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A Geometric Equality

599. *Proposed by Juan-Bosco Romero Márquez, Universidad di Valladolid, Spain.*

Let ABC be a triangle with acute angles at B and C . Let H be the foot of the perpendicular from A to BC ; let D be the foot of the perpendicular from H to AB ; let E be the foot of the perpendicular from H to AC ; let P be the foot of the perpendicular from D to BH ; and let Q be the foot of the perpendicular from E to HC . Prove that the angle at A is a right angle if and only if $\overline{AH} = \overline{DP} + \overline{EQ}$.

SOLUTION:



Let $\angle CAH = \angle EHQ = \alpha$ and $\angle BAH = \angle DHP = \beta$. Since B and C are acute angles, then α and β are also acute angles. Standard right triangle trigonometry yields

$$\sin \alpha = EQ/EH = EH/AH \text{ and } \sin \beta = DP/DH = DH/AH.$$

These equations easily yield

$$EQ = AH \cdot \sin^2 \alpha \text{ and } DP = AH \cdot \sin^2 \beta.$$

Adding gives

$$EQ + DP = AH \cdot (\sin^2 \alpha + \sin^2 \beta).$$

(\Rightarrow) If $\angle A = 90$, then $\beta = 90 - \alpha$ and $\sin^2 \beta = \sin^2(90 - \alpha) = \cos^2 \alpha$. So $EQ + DP = AH$.

(\Leftarrow) If $EQ + DP = AH$, then $\sin^2 \alpha + \sin^2 \beta = 1$ or $\sin^2 \alpha = 1 - \sin^2 \beta = \cos^2 \beta$. However, since α and β are both acute angles, this yields $\sin \alpha = \cos \beta = \sin(90 - \beta)$ and $\alpha + \beta = 90$. So, $\angle A = 90$.