Solution to Problem 611
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A Mean of Roots

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Let $f(x) = \sum_{i=1}^{n} a_i(x-x_i)^{-1}$ with $x_1 < x_2 < \cdots < x_n, n \geq 2$, and each $a_i > 0$. This function has $n-1$ real zeros, one between each consecutive pair of x_i's. Express the arithmetic mean of the zeros as a linear combination of the x_i's.

SOLUTION: The zeros of f are the roots of the equation $f(x) = 0$. Clearing fractions we see that the roots of this equation are the same as the roots of $g(x) = 0$ where

$$g(x) = \sum_{i=1}^{n} a_i \cdot \left(\prod_{\substack{j=1 \\ i \neq j}}^{n} (x - x_j) \right).$$

This is a polynomial of degree $n-1$ which we can write as $g(x) = C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0$. The arithmetic mean of the roots of $g(x) = 0$ equals

$$\sigma_1/(n-1)$$

where σ_1 is the first elementary symmetric function of the roots (the sum of the roots). It is well known that $\sigma_1 = -C_{n-2}/C_{n-1}$. An examination of g yields

$$C_{n-1} = \sum_{i=1}^{n} a_i$$

$$C_{n-2} = -\sum_{i=1}^{n} \left(a_i \sum_{\substack{j=1 \\ i \neq j}}^{n} (-x_j) \right).$$

So the arithmetic mean of the zeros of f equals

$$\frac{\sum_{i=1}^{n}\left(a_i \sum_{\substack{j=1 \\ i\neq j}}^{n}(-x_j)\right)}{(n-1)\sum_{i=1}^{n} a_i}$$