Solution to Problem 1031
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A Standing Problem

1031. *Proposed by Richard A. Gibbs, Fort Lewis College.*

There are n people, numbered consecutively, standing in a circle. First #2 sits down, then #4, #6, etc., continuing around the circle with every other standing person sitting down until just one person is left standing. What is his number? (For example, with $n = 6$, the seating order is 2,4,6,3,1 and 5 is left standing.)

SOLUTION

Let $F(n)$ be the number of the person left standing. Clearly

$$F(1) = F(2) = 1. \tag{1}$$

We claim that

$$F(n+1) = \begin{cases} F(n) + 2 & \text{if } F(n) < n, \\ 1 & \text{if } F(n) = n, \end{cases} \tag{2}$$

so that if $2^k \le n < 2^{k+1}$, then

$$F(n) = 2(n - 2^k) + 1 \tag{3}$$

Consider $n + 1$ people standing, numbered clockwise 1 through $n + 1$. After number 2 sits down, let each person standing be given a new number, beginning with person number 3 getting number 1 and continuing consecutively clockwise. Then the person left standing has a new number of $F(n)$. Hence (2) follows.

Finally, (3) follows because (1) and (2) determine F uniquely and the F given in (3) satisfies (1) and (2).

An alternate formula for F is $F(n) = 2n + 1 - 2^{1+\left\lfloor \frac{\ln n}{\ln 2} \right\rfloor}$.