

Mean and Intermediate Value Properties

1053. *Proposed by Peter Ørno, The Ohio State University.*

Let $f(x)$ be differentiable on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n , show that there exist distinct x_1, x_2, \dots, x_n such that $\sum_{i=1}^n 1/f'(x_i) = n$.

SOLUTION

More generally we show that if f is differentiable on $[a, b]$ then there exist n points $a < x_1 < x_2 < \dots < x_n < b$ such that

$$\frac{1}{n} \sum_{k=1}^n \frac{1}{f'(x_k)} = \frac{b-a}{f(b)-f(a)}.$$

Since $0 < k \leq n$ then

$$\begin{aligned} 0 &< k(f(b) - f(a)) \leq n(f(b) - f(a)) \\ n \cdot f(a) &< (n - k)f(a) + k \cdot f(b) \leq n \cdot f(b) \\ f(a) &< \frac{(n - k)f(a) + k \cdot f(b)}{n} \leq f(b) \end{aligned}$$

By the Intermediate Value Theorem, f assumes all values between $f(a)$ and $f(b)$. Divide $f(b) - f(a)$ into n equal parts. Define θ_k such that $a = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n = b$ and let

$$f(\theta_k) = \frac{(n - k)f(a) + k \cdot f(b)}{n}$$

so that $f(a) < f(\theta_k) \leq f(b)$ for all k . Now apply the Mean Value Theorem on each interval (θ_{k-1}, θ_k) .

There exist x_k on each (θ_{k-1}, θ_k) such that

$$\begin{aligned}
f'(x_k) &= \frac{f(\theta_k) - f(\theta_{k-1})}{\theta_k - \theta_{k-1}} \\
&= \frac{[(n-k)f(a) + k \cdot f(b)] - [(n-k+1)f(a) + (k-1)f(b)]}{n(\theta_k - \theta_{k-1})} \\
&= \frac{f(b) - f(a)}{n(\theta_k - \theta_{k-1})}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^n \frac{1}{f'(x_k)} &= \frac{1}{n} \sum_{k=1}^n \frac{n(\theta_k - \theta_{k-1})}{f(b) - f(a)} \\
&= \frac{1}{f(b) - f(a)} \sum_{k=1}^n (\theta_k - \theta_{k-1}) \\
&= \frac{b - a}{f(b) - f(a)}
\end{aligned}$$

The result now follows by letting $f(a) = a = 0$ and $f(b) = b = 1$