Solution to Problem 1187
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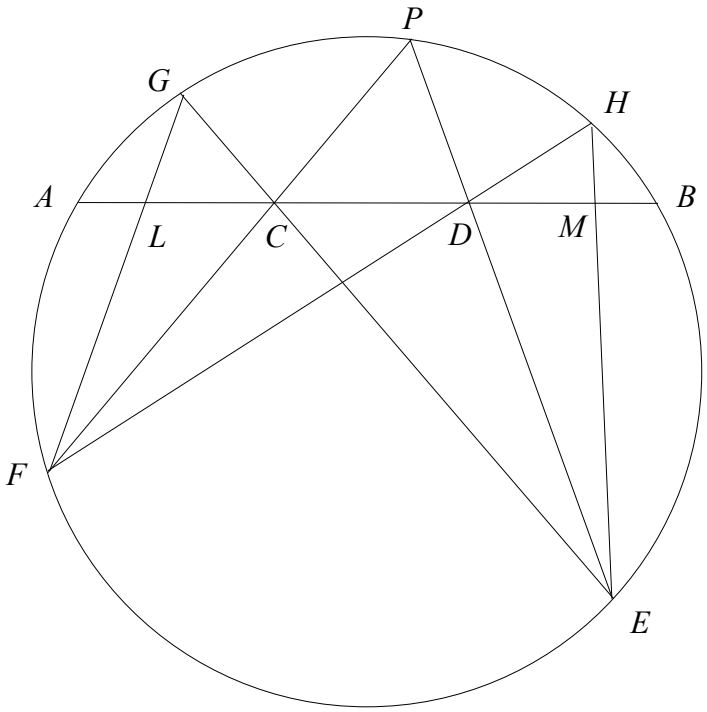
A Three-Winged Butterfly Problem

1187. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*

Let the chord AB of a circle O be trisected at C and D. Let P be any point on the circle other than A and B. Extend the lines PD and PC to intersect the circle in E and F, respectively. Extend the lines EC and FD to intersect the circle in G and H, respectively. Let GF and HE intersect AB in L and M, respectively. Prove that $AL = BM$.



SOLUTION

In *The Two Year College Mathematics Journal*, vol. 14 (1983), p. 3, Ross Honsberger states a lemma shown to him by his friend Professor Haruki.

LEMMA: Suppose AB and FE are nonintersecting chords in a circle, and that Q is a variable point on the arc AB remote from F and E. Then for each position of Q, the lines QF

and QE cut AB into three segments of lengths x, y, and z (in order) such that xy/z in independent of Q.

Applying this lemma twice (with Q taken as G and H), we find that

$$\frac{AL \cdot CB}{LC} = \frac{AD \cdot MB}{DM}$$

Since $AC = DB$, it follows that $CB = AD$. Reducing and inverting the proportion gives $LC/AL = DM/MB$. Adding 1 and reducing yields $AL = MB$.