Solution to Problem 1190
The Mathematics Magazine

A Reversed Digit Problem

1190. *Proposed by Robert L. Bernhardt, East Carolina University.*

(a) If abc is a three-digit number in base ten, where $a > c$, and if $def = abc - cba$ is always considered a three-digit number (even when $d = 0$), then it is known that $def + fed = 1089$. Generalize this result to any base $k > 1$.

(b) For which base(s) is $def + fed$ a perfect square?

SOLUTION

(a) Let abc be any three-digit number in base k with $0 \leq a, b, c < k$ and $a > c$. Then abc has a value of $ak^2 + bk + c$ and cba has a value of $ck^2 + bk + a$. Since $def = abc - cba$ then def has a value of

$$(a - c)k^2 + 0k + (c - a).$$

Since $c - a$ is negative, it cannot represent a digit in base k. To express it as a digit, borrow 1 from the "hundreds" place. That is, the value of def can be expressed as $(a-c-1)k^2+(k-1)k+(c+k-a)$. Therefore fed has a value of $(c+k-a)k^2+(k-1)k+(a-c-1)$. Adding these two expressions, we find that $def + fed$ has a value of

$$(k - 1)k^2 + 2(k - 1)k + (k - 1)$$

which simplifies to $(k - 1)(k + 1)^2$.

(b) Clearly this expression is a perfect square if $k = n^2 + 1$. (i.e. $k = 2, 5, 10, 17, \ldots$)