Solution to Problem 1290
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# A Combinatorial Sum

**1290.** *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Canada.*

For positive integers $n$ and $r$, let $\left\langle {n \atop k} \right\rangle = \binom{n+r-1}{r}$. Find a closed form expression for

$$\sum_{r=1}^{k} r \left\langle {n \atop k} \right\rangle,$$

where $k$ denotes a positive integer.

SOLUTION

$$\begin{aligned}
\sum_{r=1}^{k} r \left\langle {n \atop k} \right\rangle &= \sum_{r=1}^{k} r \binom{n+r-1}{r} \\
&= \sum_{r=1}^{k} \frac{r(n+r-1)!}{r!(n-1)!} \\
&= \sum_{r=1}^{k} \frac{n(n+r-1)!}{(r-1)!n!} \\
&= n \sum_{r=1}^{k} \binom{n+r-1}{n} \\
&= n \sum_{r=0}^{k-1} \binom{n+r}{n}
\end{aligned}$$

This last sum is the familiar "hockey stick" formula which equals

$$n\binom{n+k}{n+1}.$$