Solution to Problem 1298
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Circumscribable quadrangle

1298. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA respectively. Let $a = |AB|, b = |BC|, c = |CD|, d = |DA|$ and $p = |QS|, q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}$$

SOLUTION

Without loss of generality, let the radius of the circle equal one. Let the center of the circle be O. Draw $OA, OB, OC, OD, OP, OQ, OR, OS$. Let $\angle AOS = \angle AOP = \alpha, \angle BOP = \angle BOQ = \beta, \angle COQ = \angle COR = \gamma, \angle DOR = \angle DOS = \delta$.

Basic right triangle trigonometry yields:

$$\begin{array}{rcl} a = \tan\alpha + \tan\beta & & c = \tan\gamma + \tan\delta \\ b = \tan\beta + \tan\gamma & & d = \tan\alpha + \tan\delta \\ p = 2\sin(\alpha+\beta) & \Rightarrow & p^2 = 4\sin^2(\alpha+\beta) \\ q = 2\sin(\alpha+\delta) & \Rightarrow & q^2 = 4\sin^2(\alpha+\delta) \end{array}$$

Also, $2\alpha + 2\beta + 2\gamma + 2\delta = 360° \Rightarrow \alpha + \beta + \gamma + \delta = 180°$.

So, $\sin(\alpha+\beta) = \sin(\gamma+\delta)$ and $\sin(\alpha+\delta) = \sin(\beta+\gamma)$.

Now, $a = \tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$ and in the same way $c = \tan\gamma + \tan\delta = \dfrac{\sin(\gamma+\delta)}{\cos\gamma\cos\delta}$.

Therefore

$$\begin{aligned}\frac{ac}{p^2} &= \frac{\sin(\alpha+\beta)\sin(\gamma+\delta)}{4\sin^2(\alpha+\beta)\cos\alpha\cos\beta\cos\gamma\cos\delta} \\ &= \frac{1}{4\cos\alpha\cos\beta\cos\gamma\cos\delta}\end{aligned}$$

In a similar way, $\dfrac{bd}{q^2} = \dfrac{1}{4\cos\alpha\cos\beta\cos\gamma\cos\delta}$.