Solution to Problem 1354
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Quadrilateral subdivision

1354. *Proposed by Frank Schmidt, Bryn Mawr College, Bryn Mawr, Pennsylvania, and Rodica Simion, George Washington University, Washington, DC.*

Let $ABCD$ be a convex quadrilateral in the plane with trisection points joined as in the figure to form nine smaller quadrilaterals.

(a) Show that the area of $A'B'C'D'$ is one-ninth the area of $ABCD$.

(b) Give necessary and sufficient conditions so that all nine quadrilaterals have equal area.

SOLUTION

(a) Part (a) of this problem was solved by B. Greenberg, *That Area Problem*, The Mathematics Teacher, Vol. 64, 1971, pp. 79-80. A generalization and some related results appeared as Problem E2423, The American Mathematical Monthly, Vol. 81, No. 6, June-July 1974, pp. 666-668, solved by Donald Batman and Murray Klamkin.

(b) To prove part (b) we will use the notation of problem E2423 referred to above. Actually more can be proved. Let one pair of opposite sides be cut into m equal segments and the other pair of opposite sides be cut into n equal segments. According to Batman and Klamkin, the area of any one of the mn smaller quadrilaterals is given by

$$|Q(r,s)| = \frac{A}{mn(p+q+2)}\left(2 + \frac{(2s+1)p}{n} + \frac{(2r+1)q}{m}\right)$$

Equating any two such smaller quadrilaterals yields

$$\frac{A}{mn(p+q+2)}\left(2 + \frac{(2s_1+1)p}{n} + \frac{(2r_1+1)q}{m}\right) = \frac{A}{mn(p+q+2)}\left(2 + \frac{(2s_2+1)p}{n} + \frac{(2r_2+1)q}{m}\right)$$

This equation quickly simplifies to $(p/n)(s_1 - s_2) + (p/m)(r_1 - r_2) = 0$. Since m and n are fixed, r_1 and r_2 vary from 0 to m, s_1 and s_2 vary from 0 to n, then this equation will be identically zero if and only if $p = q = 0$. However this implies that $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{OC}$ and the quadrilateral must be a parallelogram.